A NEW TREATMENT OF SINGULARITIES IN GALERKIN BOUNDARY INTEGRAL EQUATIONS

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1 Abstract

One important issue in boundary element methods is the evaluation of singular and almost singular integrals. The present method consists in recursively reducing the dimension of the integration domain, so as to obtain a linear combination of one-dimensional regular integrals, with coefficients depending on the relative positions of the panels. Moreover explicit formulas can be derived for these integrals.

Our method relies upon the homogeneity of the integrand [4] and applies to the singular part of the Green function. It is not the only one devoted to the evaluation of singular integrals [2] but has appealing features in terms of precision, efficiency and ability to apply to almost singular cases (Figure 1).

We only present here two 3D results with constant densities, but the method applies as well to 2D, single and double layers, linear densities (work in progress) or even volume integral equations.

2 Introduction

Variational boundary element methods lead to the evaluation of such integrals as

$$\int_{S\times T} G(x,y)\, v(x)\, w(y)\, dxdy \text{ or}$$
$$\int_{S\times T} \frac{\partial}{\partial n_y} G(x,y)\, v(x)\, w(y)\, dxdy$$

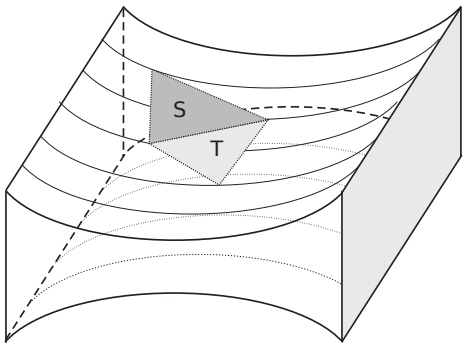
where v and w are basis functions, G the Green kernel, singular on the diagonal, and S and T plane polygons from the discretization of the boundary. For Helmholtz or Maxwell equations, the Green function expands as follows :

$$\begin{aligned} G(x,y) &= -\frac{1}{4\pi}\frac{1}{\|x-y\|} + H(x,y) \\ \nabla_y G(x,y) &= -\frac{1}{4\pi}\frac{x-y}{\|x-y\|^3} - \frac{k^2}{8\pi}\frac{x-y}{\|x-y\|} \\ &+ K(x,y) \end{aligned}$$

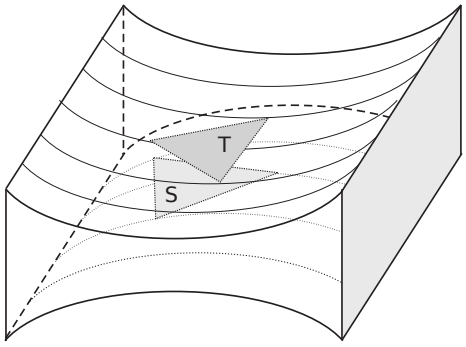
where H and K are regular functions. The integration of the regular parts involves only classical numerical quadrature, whereas the singular ones

$$\begin{aligned} I &= \int_{S\times T} \frac{1}{\|x-y\|} dxdy \\ J_\zeta &= \int_{S\times T} \frac{x-y}{\|x-y\|}\frac{}{^{1+\zeta}} dxdy \end{aligned} \quad (1)$$

are subject to our method.



Logically close triangles



Logically distant triangles

Figure 1: Geometrically close triangles

3 Basic formulas

Let $f(x,d) : \Omega \subset \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$ a positively homogeneous function of degree q, where d will be referred to as the parameter. We denote

$$I(d) = \int_\Omega f(z,d)\, dz \quad (2)$$

which by Euler's formula and Green's theorem satisfies the following differential equation :

$$(q+n)\, I(d) = dI'(d) + \int_{\partial\Omega} (\overrightarrow{z}\,|\,\overrightarrow{\nu}\,)\, f(z,d)\, d\gamma_z$$

where $\overrightarrow{\nu}$ is the exterior normal to Ω and $(\cdot|\cdot)$ is the scalar inner product.

Provided $d^{-(q+n)} \int_\Omega f(z,d)\, dz \to 0$ as $d \to +\infty$ one obtains

$$I(d) = d^{q+n} \int_{\partial\Omega} (\overrightarrow{z}\,|\,\overrightarrow{\nu}\,) \int_d^{+\infty} \frac{f(z,t)}{t^{q+n+1}} dt d\gamma_z. \quad (3)$$

When $f(z,d)$ does not depend on d and $q+n \neq 0$ then

$$I = \frac{1}{q+n} \int_{\partial\Omega} (\overrightarrow{z}\,|\,\overrightarrow{\nu}\,)\, f(z)\, d\gamma_z. \quad (4)$$

As far as the inner integral in (3) can be explicitely evaluated, both formulas reduce an n-dimensional integral to an $(n-1)$ one. When Ω is an n-dimensional polyhedron, $(\overrightarrow{z}\,|\,\overrightarrow{\nu}\,)$ is constant on each $(n-1)$-face of Ω, a simplification of crucial importance in the sequel.

4 The reduction process

Despite some (possibly) lengthy calculations, the principle is rather straightforward and the method is quite flexible, leading to the reduction of 4-D integrals to a linear combination of 1-D regular integrals which can be numerically or even explicitely evaluated. Depending on the relative positions of S and T, we are lead to use formula (4) or (3) or both (Figure 2).

As an example, we give now a detailed presentation of the reduction process in the simple but significant case of the self-influence coefficient

$$I = \int_{S\times S} \frac{1}{\|x-y\|} dxdy$$

which leads to a surprisingly simple formula.

Let A_i be some vertex of the triangle, α_i the opposite side, γ_i their mutual distance and λ_i the exterior normal along α_i (see notations on Figure 3).

As homogeneity is not invariant by change of variables, we must choose carefully the origin of the coordinates in order to apply formulas (4) and (3). Accordingly, we take the origin in the plane of the triangle. As $\partial(S\times S) = (\partial S \times S) \cup (S \times \partial S)$, $q=-1$ and $q+n=3$, by (4) one obtains

$$I = \frac{2}{3} \sum_{i=1,3} \int_{\alpha_i\times S} \left(\overrightarrow{x}\,\middle|\,\overrightarrow{\lambda_i}\right) \frac{1}{\|x-y\|} dxdy$$

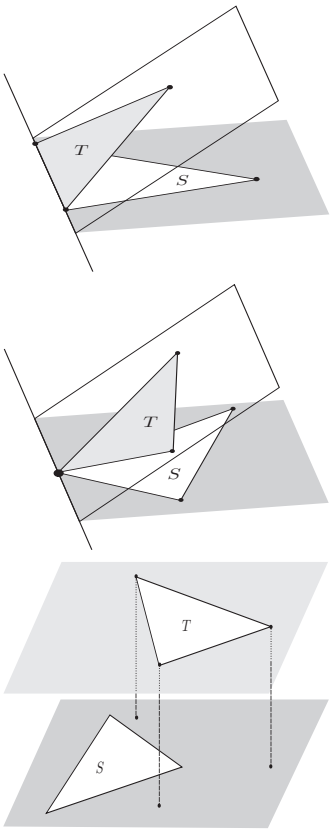


Figure 2: Various geometric configurations

Moreover, with A_i as origin and $x \in \alpha_i \times S$, $\left(\overrightarrow{x}\,\middle|\,\overrightarrow{\lambda_j}\right) = 0$, $j \neq i$ and $\left(\overrightarrow{x}\,\middle|\,\overrightarrow{\lambda_i}\right) = \gamma_i$, from which e.g.

$$I = \frac{2\gamma_1}{3} \int_{\alpha_1\times S} \frac{1}{\|x-y\|} dxdy \quad (5)$$

The next step in the reduction process of (5) consists in choosing one end, say A_2, of α_1 as the new origin, leading to

$$\begin{aligned} I &= \tfrac{1}{3}\gamma_1 |\alpha_1| \int_S \tfrac{1}{\|A_3-y\|} dy \\ &+ \tfrac{1}{3}\gamma_1\gamma_2 \int_{\alpha_1\times\alpha_2} \tfrac{1}{\|x-y\|} dxdy \end{aligned}$$

The final step requires to choose with A_3 as the new origin. Setting

$$R(A_i,\alpha_i) = \int_{\alpha_i} \frac{1}{\|A_i-y\|} dy$$

one has

$$\int_S \frac{1}{\|A_3-y\|} dy = \gamma_3 R(A_3,\alpha_3)$$

$$\int_{\alpha_1\times\alpha_2} \frac{1}{\|x-y\|} dxdy = |\alpha_1|\, R(A_1,\alpha_1) + |\alpha_2|\, R(A_2,\alpha_2)$$

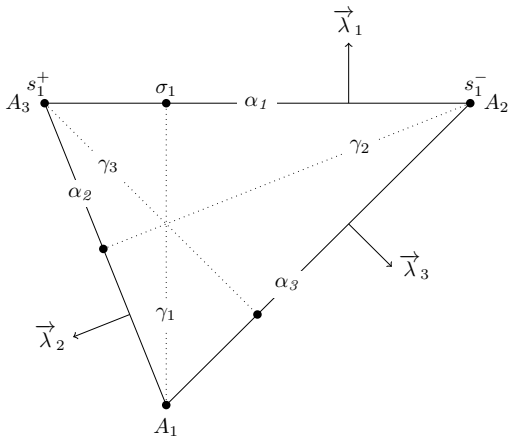


Figure 3: Notations

The final formula involves only regular one-dimensional integrals ; it reads as

$$\boxed{I = \frac{2\,|S|}{3} \sum_{i=1,3} \gamma_i R\left(A_i, \alpha_i\right)} \tag{6}$$

It can be evaluated by straightforward numerical quadrature or even explicitely as

$$R\left(A_i, \alpha_i\right) = \arg\sinh \frac{s_i^+ - \sigma_i}{\gamma_i} - \arg\sinh \frac{s_i^- - \sigma_i}{\gamma_i}$$

(see notations on figure 3).

The complexity of the final formula depends heavily on the relative positions of S and T which can be described with only 3 parameters in the case of self influence coefficients and 12 in the general one. More important is the fact that one needs formula (3) in the place of (4) when S and T belong to parallel planes or when α and β do not meet in the evaluation of such terms as

$$\int_{\alpha\times\beta} \frac{1}{\|x-y\|} dxdy$$

The main difficulty arises when α and β are almost parallel, as the distances of their intersection to the ends of α and β appear as coefficients in the formula, resulting in severe cancellation between large terms. In this case, depending on the expected precision, one must derive a criterion for switching to the parallel case, where the formula does not involve that drawback.

4.1 Comparison with alternative methods

As a first numerical result, we compare formula (6) referred to as 'exact result', with two other methods. Tables 1 and 2 show their relative precision with respect to the number of integration points. The first one is a direct integration of the singular integrand [3] (referred to as Cubpack[1]), the second one is the coordinate transformation method of Sauter and Schwab [2] which involves a regular integral over a 4D hypercube. The quadrature has been performed via a tensor product of Gauss-Legendre formulas.

Method	**Integration Points**	**Relative error**
Sauter-Schwab	1	$2.017e-01$
	81	$1.042e-03$
	625	$2.540e-05$
	1296	$4.025e-06$
Cubpack	37	$2.339e-04$
	185	$5.966e-05$
	1503	$4.589e-06$

Table 1: Triangle $\{(0,0),(1,0),(1,1)\}$
Exact result : 1.0030658847731824

Method	**Integration Points**	**Relative error**
Sauter-Schwab	1	$4.085e-01$
	81	$2.486e-02$
	625	$5.006e-03$
	1296	$3.129e-03$
Cubpack	37	$2.984e-04$
	185	$7.621e-05$
	1369	$8.131e-05$

Table 2: Triangle $\{(0,0),(4,0),(0.5,0.5)\}$
Exact result : $I = 2.265846111074699$

4.2 Superposed triangles

One interesting case is the computation of the influence between superposed triangles, subject to a constant density of double layer potential. With formula (3) instead of (4), one obtains (see (1))

$$\boxed{J_2 = 4|S|\overrightarrow{v} \sum_{i=1,3} \gamma_i \mathfrak{R}(A_i, \alpha_i)} \tag{7}$$

[1]With the contribution of M. Fares, Cerfacs, France for the implementation.

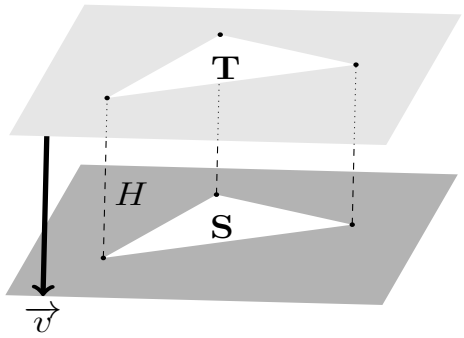


Figure 4: Superposed triangles

where $\overrightarrow{v}$ is defined on Figure 4 and

$$\begin{aligned}\mathfrak{R}(A_k,\alpha_k) &= \left[s\,\frac{\sqrt{H^2+\gamma_k^2+s^2}-H}{2\gamma_k^2(\gamma_k^2+s^2)} - \frac{s\,\arg\sinh\left(\frac{\sqrt{\gamma_k^2+s^2}}{H}\right)}{\gamma_k^2\sqrt{\gamma_k^2+s^2}}\right.\\ &+\frac{1}{\gamma_k^2}\arg\sinh\left(\frac{s}{\sqrt{H^2+\gamma_k^2}}\right) - \frac{(\gamma_k^2-H^2)\,\pi\,\operatorname{sgn} s}{4H\gamma_k^3}\\ &+\frac{\gamma_k^2-H^2}{2H\gamma_k^3}\mathrm{Im}\left\{\arg\tanh\left(\frac{H^2+\gamma_k^2+i\gamma_k s}{H\sqrt{H^2+\gamma_k^2+s^2}}\right)\right\}\\ &\left.+\frac{\gamma_k^2-H^2}{2H\gamma_k^3}\arg\tan\left(\frac{s}{\gamma_k}\right)\right]_{s^--\sigma}^{s^+-\sigma}\end{aligned}$$

It is worth to notice that the solid angle formula shows that $\left(J_2\left|\frac{\overrightarrow{v}}{\|\overrightarrow{v}\|}\right.\right)$ goes to $2|S|\pi$ when the distance H between the two planes vanishes, a situation considered as particularly difficult from the numerical point of view. Figure 5 shows that formula (7) does admit this behaviour with a high accuracy.

5 Conclusion

The results we have presented are only a small part of all those available which allow to take into account the various geometric situations. The method is very flexible in that it transforms the original integral in a combination of 3D, 2D or 1D integrals or in an explicit formula, as required. A high degree of accuracy can be obtained, even in the case of almost singular integrands. The major drawback lies in taking into account the cases of almost parallelism, which can be treated through the approximation of exact parallelism.

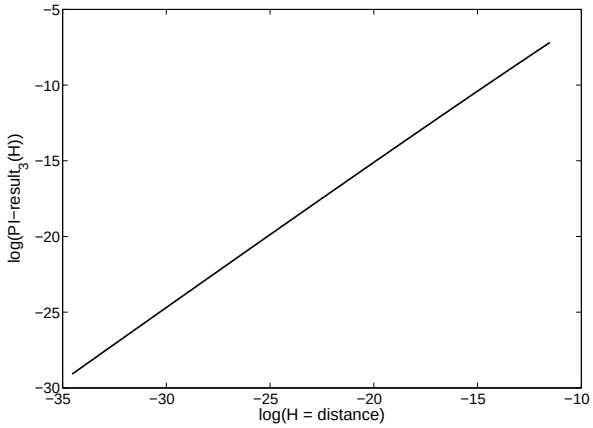


Figure 5: $\pi - \left(J_2(H)|\frac{\vec{v}}{\|\vec{v}\|}\right)$ when $H \to 0$ in log-log scale for triangle $\{(0,0),(1,0),(0,1)\}$.

References

[1] M. Lenoir, "Influence coefficients for variational integral equations.", C. R., Math., Acad. Sci. Paris, vol. 343, pp. 561-564, 2000.

[2] S. Sauter and C. Schwab, Boundary Element Methods, Springer-Verlag Berlin, 2010.

[3] R. Cools and A. Haegemans, "Algorithm 824: CUBPACK: A package for automatic cubature; framework description.", ACM Trans. Math. Softw., vol. 3, pp. 287-296, 2003.

[4] Rosen, D. and Cormack, D.E., "Singular and near singular integrals in the BEM: A global approach.", SIAM Journal on Applied Mathematics, vol. 53, pp. 340-357, 1993.