A NEW TREATMENT OF SINGULAR INTEGRALS IN GALERKIN BOUNDARY INTEGRAL EQUATIONS

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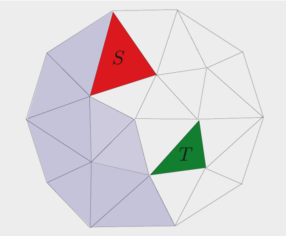
Outline

What do we want ?

We want to evaluate

$$\int_{\Gamma\times\Gamma} q(y)G(x,y)\bar{t}(x)d\gamma_x d\gamma_y \text{ and } \int_{\Gamma\times\Gamma} p(y)\frac{\partial G(x,y)}{\partial n_y}\bar{s}(x)d\gamma_x d\gamma_y,$$

We discretize Γ with plane polygons (usually triangles)



$$\int_{S\times T} \phi_i(x)G(x,y)\phi_j(y)d\gamma_x d\gamma_y$$

$$\int_{S\times T} \psi_i(x)\frac{\partial G(x,y)}{\partial n_y}\psi_j(y)d\gamma_x d\gamma_y$$

Singular Integrals

Green kernel and its gradient

$$G(x, y) = -\frac{e^{ik\|x-y\|}}{4\pi\|x-y\|} = -\frac{1}{4\pi}\frac{1}{\|x-y\|} + H(\|x-y\|)$$

$$\nabla G(x, y) = -\frac{1}{4\pi}\frac{x-y}{\|x-y\|^3} - \frac{k^2}{8\pi}\frac{x-y}{\|x-y\|} + (x-y)K(\|x-y\|)$$

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Singular integrals

$$\int_{S\times T}\frac{1}{\|x-y\|}dxdy\,;\ \int_{S\times T}\frac{x-y}{\|x-y\|^3}dxdy\,;\ \int_{S\times T}\frac{x-y}{\|x-y\|}dxdy$$

The issues

Everyone knows that it is essential to calculate these integrals very carrefuly and with high accuracy.

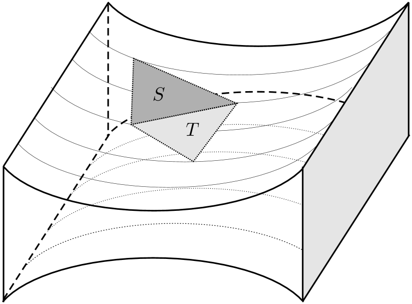
References

• M. G. Duffy. *Quadrature over a pyramid or cube of integrands with a singularity at a vertex*, SIAM J. Numer. Anal., 1982

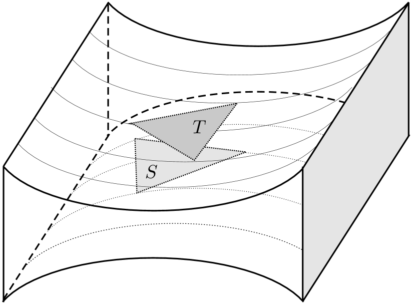
• S. Nintcheu Fata, *Semi-analytic treatment of nearly-singular Galerkin surface integrals*, Journal of computational and applied mathematics, 2009

• S. Sauter and C. Schwab, *Boundary Element Methods*, Springer Series in Computational Mathematics, 2010

Singularities



Singular case



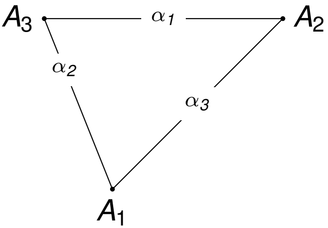
Almost singular case

Lenoir M., *Influence coefficients for variational integral equations*, Comptes Rendus Mathematique, 2006

First Case : The self-influence

We want to evaluate **exactly** :

$$I = \int_{S \times S} \frac{dxdy}{\|x - y\|}$$



We define,

$$\sigma_i = \frac{\|P_i - A_i^-\|}{|\alpha_i|}$$

We will prove that we obtain

$$I = \frac{2\,|S|}{3} \sum_{i=1}^{3} \gamma_i \left(\operatorname{arg\,sinh}\left(\frac{|\alpha_i| - \sigma_i}{\gamma_i} \right) + \operatorname{arg\,sinh}\left(\frac{\sigma_i}{\gamma_i} \right) \right).$$

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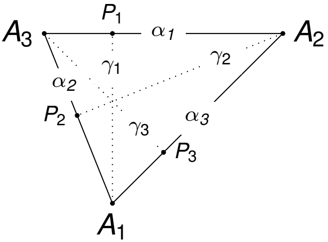

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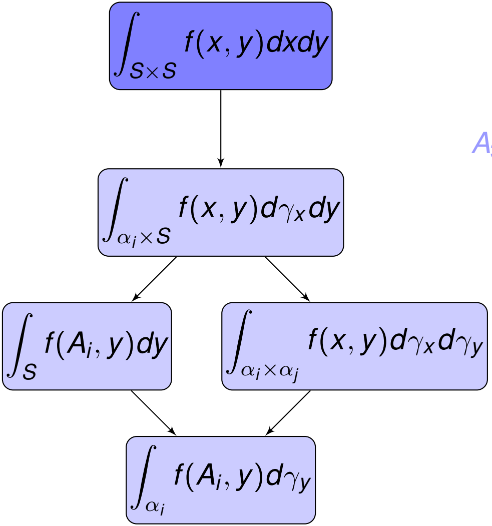
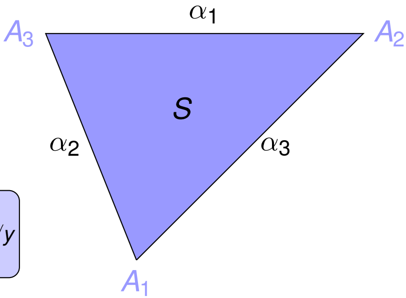
The Reduction Process





The Reduction Process

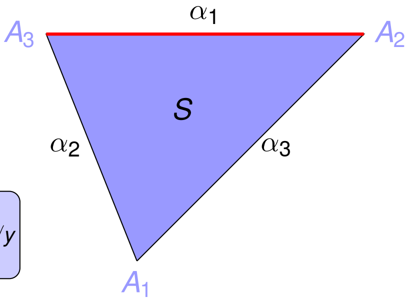
$$\int_{S\times S} f(x,y)\,dx\,dy$$

$$\int_{\alpha_i\times S} f(x,y)\,d\gamma_x\,dy$$

$$\int_{S} f(A_i,y)\,dy$$

$$\int_{\alpha_i\times\alpha_j} f(x,y)\,d\gamma_x\,d\gamma_y$$

$$\int_{\alpha_i} f(A_i,y)\,d\gamma_y$$



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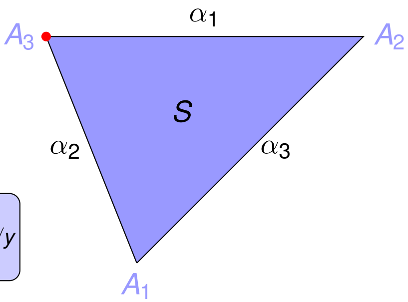
$$\int_{S} f(A_i,y)\,dy \qquad \int_{\alpha_i\times\alpha_j} f(x,y)\,d\gamma_x\,d\gamma_y$$

$$\int_{\alpha_i} f(A_i,y)\,d\gamma_y$$



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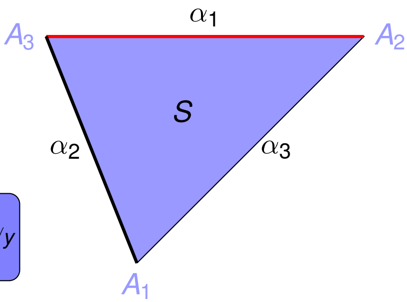

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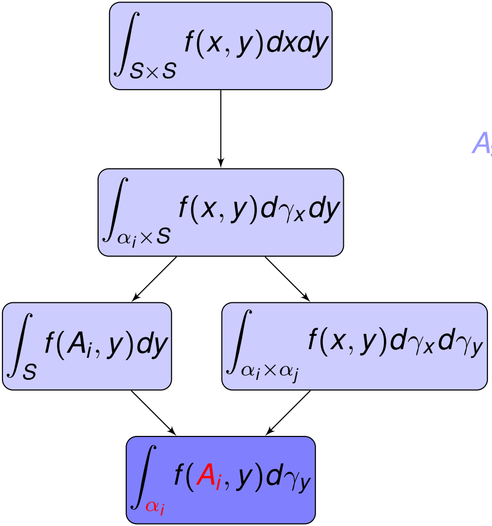
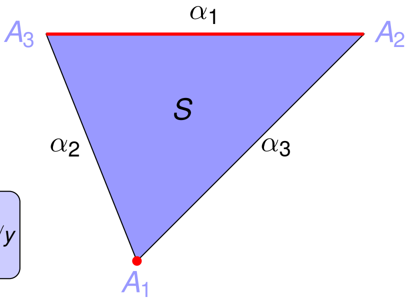




The First Formula

Suppose

- $\Omega \subset \mathbb{R}^n$
- $f : \Omega \to \mathbb{R}$ is a homogeneous function of degre r :

$$f(\lambda z) = \lambda^r f(z), \forall \lambda > 0$$

Then

$$\int_\Omega f(z)dz = \frac{1}{r+n} \int_{\partial\Omega} (z|\overrightarrow{\nu})\, f(z) d\gamma_z$$

Proof.

By integration of the Euler's theorem on homogeneous functions □

Rosen D., Cormack D. E., *Singular and Near Singular Integrals in the Bem: A Global Approach*, SIAM J.A.M., 1993

The First Formula

In our case

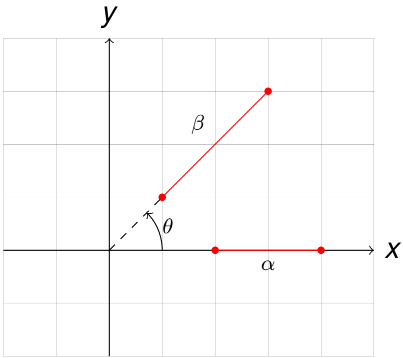
We use

$$\int_{\Omega} f(z)dz = \frac{1}{r+n} \int_{\partial\Omega} (z|\overrightarrow{\nu})\, f(z) d\gamma_z$$

With

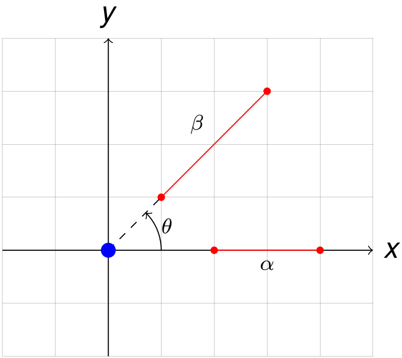
- $\Omega = S \times S \Longrightarrow n = 4$
- $z = (x, y) \in S \times S$
- $f(x, y) = \dfrac{1}{\|x - y\|} \Longrightarrow r = -1$
- Ω is a polyhedron so $\overrightarrow{\nu}$ is piecewise constant.

The choice of the origin



$$\int_{\alpha\times\beta} \frac{d\gamma_x \, d\gamma_y}{\|x-y\|}$$

The choice of the origin



$$x_1(s) = s$$
$$x_2(s) = 0$$
$$y_1(t) = t\cos(\theta)$$
$$y_2(t) = t\sin(\theta)$$

$$\int_{\alpha\times\beta} \frac{d\gamma_x\, d\gamma_y}{\|x-y\|} = \int_s \int_t \frac{ds\, dt}{\sqrt{(s - t\cos(\theta))^2 + (t\sin(\theta))^2}}$$

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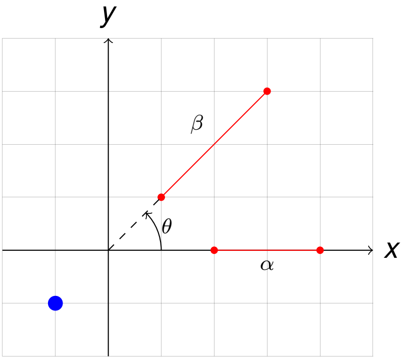

$$x_1(s) = s$$
$$x_2(s) = \mathbf{1}$$
$$y_1(t) = t\cos(\theta)$$
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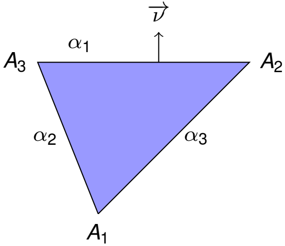
$$\int_{\alpha\times\beta} \frac{d\gamma_x\, d\gamma_y}{\|x-y\|} = \int_s \int_t \frac{ds\, dt}{\sqrt{(s - t\cos(\theta))^2 + (\mathbf{1} - t\sin(\theta))^2}}$$

The reduction process

The self-influence

$$I = \int_{S\times S} \frac{dxdy}{\|x-y\|}$$



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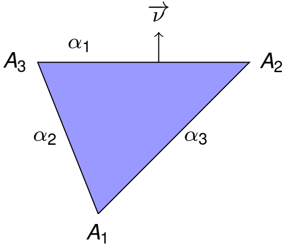

$$\int_{\Omega} f(z)dz = \frac{1}{r+n}\int_{\partial\Omega}(z|\overrightarrow{\nu})f(z)d\gamma_z$$

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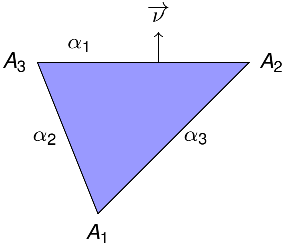

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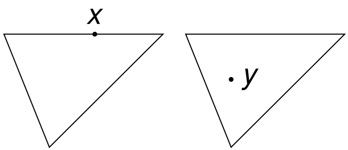

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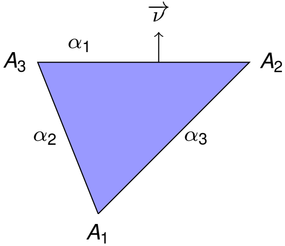

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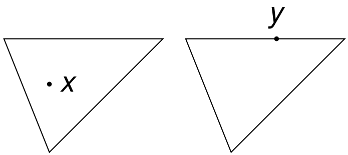

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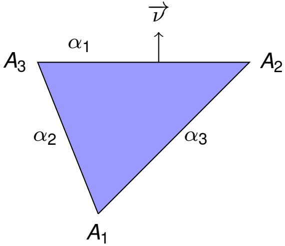
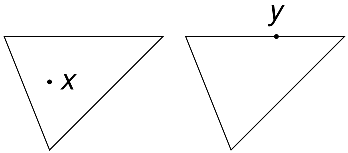


$$I = \frac{1}{3}\int_{\partial S\times S} \frac{(x|\overrightarrow{\nu})}{\|x-y\|} d\gamma_x dy + \frac{1}{3}\int_{S\times \partial S} \frac{(y|\overrightarrow{\nu})}{\|x-y\|} dx d\gamma_y$$



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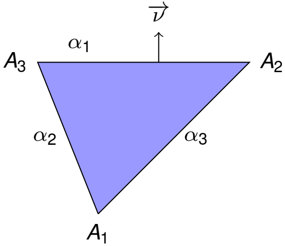
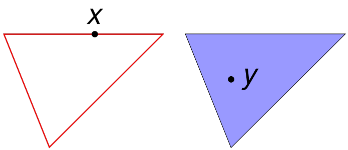


$$I = \frac{2}{3} \int_{\partial S \times S} \frac{(x|\overrightarrow{\nu})}{\|x - y\|} dxdy$$



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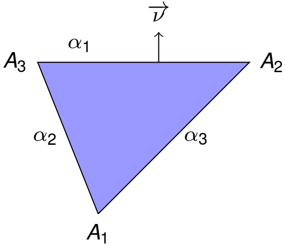

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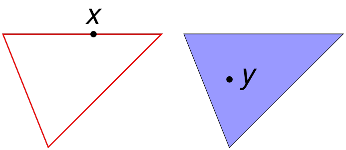

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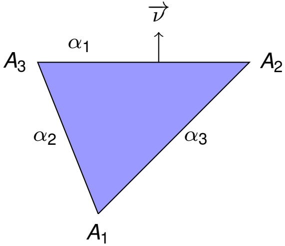
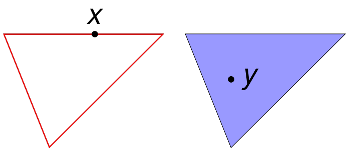


$$\partial S = \bigcup_{i=1..3} \alpha_i$$



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$$I = \int_{S\times S} \frac{dxdy}{\|x-y\|}$$

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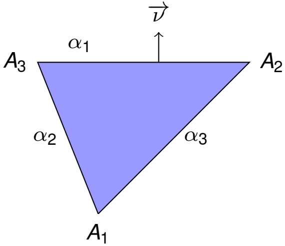

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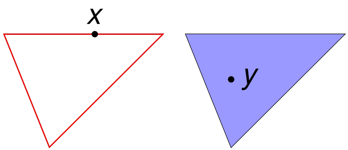

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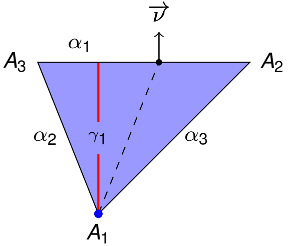

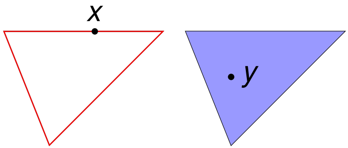

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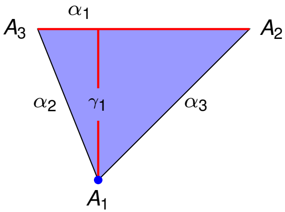
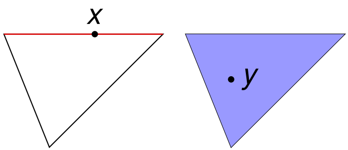
$$I = \int_{S\times S} \frac{dxdy}{\|x-y\|}$$

$$I = \frac{2}{3}\,\gamma_1 \int_{\alpha_1\times S} \frac{dxdy}{\|x-y\|}$$





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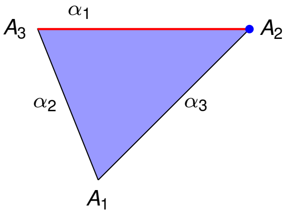

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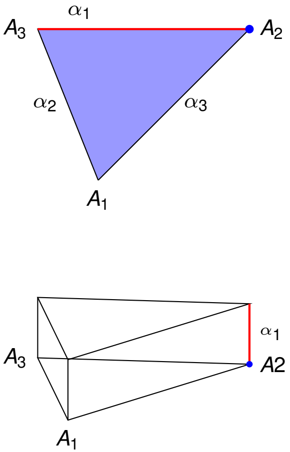

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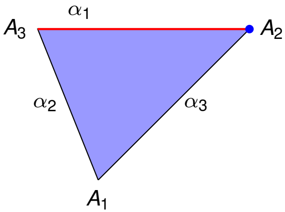
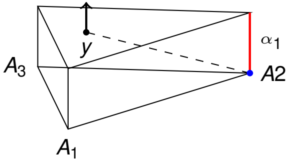


$$\partial\left(\alpha_1 \times S\right) = \left(\partial\alpha_1 \times S\right) \cup \left(\alpha_1 \times \partial S\right)$$



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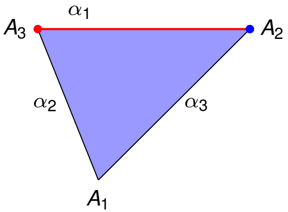

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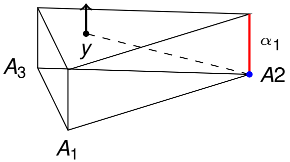

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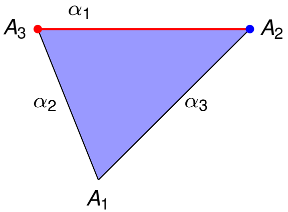

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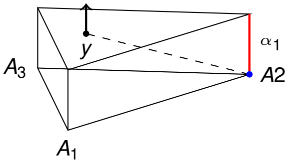

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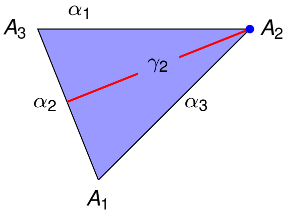
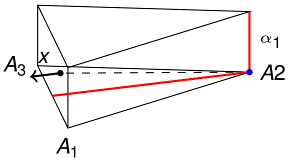


$$\partial\left(\alpha_1 \times S\right) = \left(A_3 \times S\right) \cup \left(\alpha_1 \times \partial S\right)$$

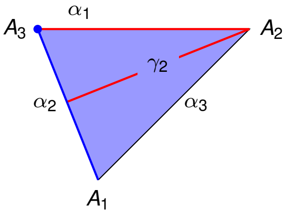


The reduction process

The self-influence

$$I = \frac{2}{3}\,\gamma_1 \int_{\alpha_1 \times S} \frac{dxdy}{\|x-y\|}$$

$$I = \frac{2}{3}\,\gamma_1 \frac{|\alpha_1|}{2} \int_S \frac{dy}{\|A_3 - y\|} + \frac{2}{3}\,\gamma_1 \frac{\gamma_2}{2} \int_{\alpha_1 \times \alpha_2} \frac{d\gamma_x d\gamma_y}{\|x-y\|}$$



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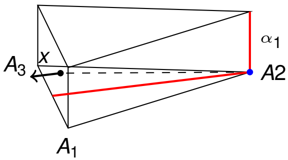
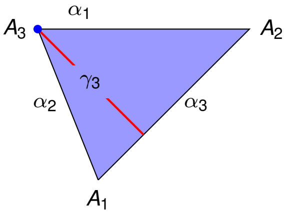


The reduction process

The self-influence

$$I = \frac{2}{3}\,|S| \int_S \frac{dy}{\|A_3 - y\|} + \frac{1}{3}\,\gamma_1\gamma_2 \int_{\alpha_1\times\alpha_2} \frac{d\gamma_x d\gamma_y}{\|x-y\|}$$



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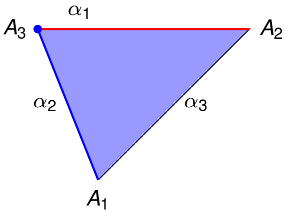
$$
\begin{aligned}
I &= \frac{2}{3}|S|\gamma_3 \int_{\alpha_3} \frac{d\gamma_y}{\|A_3 - y\|} \\
&+ \frac{1}{3}\gamma_1\gamma_2 \int_{\alpha_1\times\alpha_2} \frac{d\gamma_x d\gamma_y}{\|x - y\|}
\end{aligned}
$$



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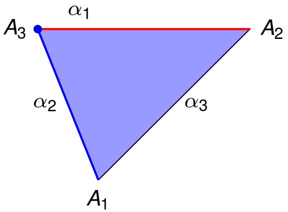
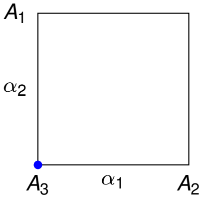


$$\partial(\alpha_1 \times \alpha_2) = (\{A_3, A_2\} \times \alpha_2) \cup (\alpha_1 \times \{A_3, A_1\})$$



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\end{aligned}
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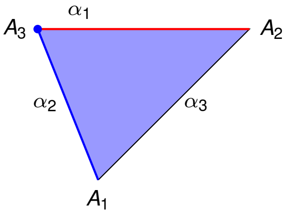

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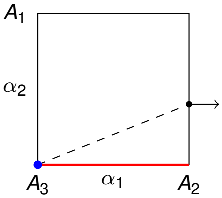

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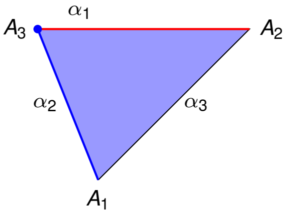
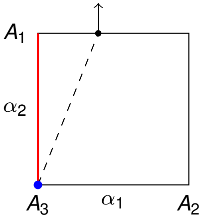
\end{aligned}
$$

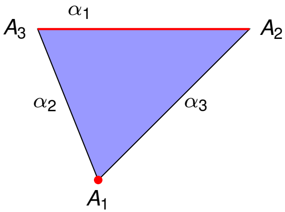




The reduction process

The self-influence

$$I = \frac{2|S|}{3} \sum_{i=1..3} \gamma_i \int_{\alpha_i} \frac{dx}{\|A_i - x\|}$$



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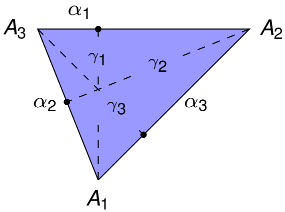
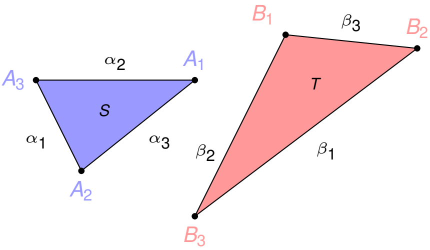


$$I = \frac{2\,|S|}{3} \sum_{i=1}^{3} \gamma_i \left(\arg\sinh\left(\frac{|\alpha_i| - \sigma_i}{\gamma_i}\right) + \arg\sinh\left(\frac{\sigma_i}{\gamma_i}\right) \right)$$

Second case : In the same plane

$$I(S,S) = \frac{2|S|}{3} \sum_{i=1..3} \gamma_i R(A_i, \alpha_i)$$

with $$R(A_i, \alpha_i) = \int_{\alpha_i} \frac{dx}{\|x - A_i\|}$$



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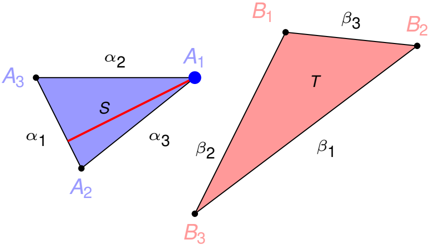

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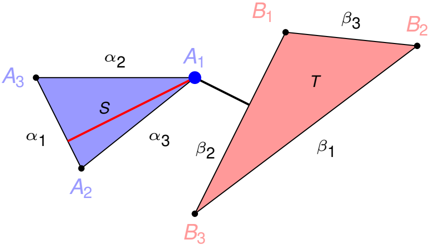

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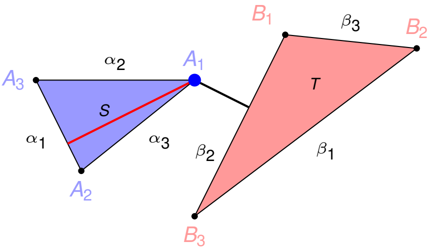

$$I(S,T) = \frac{\gamma_1}{3} \int_{\alpha_1 \times T} \frac{d\gamma_x dy}{\|x - y\|} + \sum_{j=1..3} \frac{\delta_j(A_1)}{3} \int_{S \times \beta_j} \frac{dx d\gamma_y}{\|x - y\|}$$

Second case : In the same plane

$$I(S,S) = \frac{2|S|}{3} \sum_{i=1..3} \gamma_i R(A_i, \alpha_i)$$

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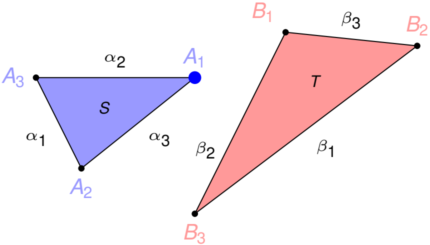

$$I(S,T) = \sum_{i=1..3} \sum_{j=1..3} \mathcal{C}_{i,j} R(A_i, \beta_j) + \mathcal{D}_{i,j} R(B_j, \alpha_i)$$

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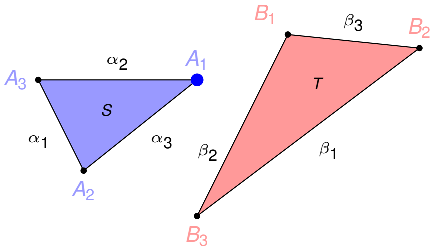

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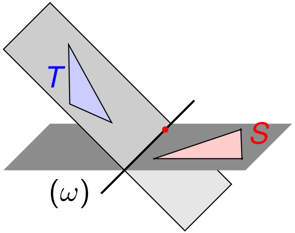
$$\mathcal{C}_{1,2} = \frac{\delta_2(A_1)}{6} \left[\gamma_2(B_3) \left(A_1 - I_{\alpha_2,\beta_2} | \frac{A_1 - A_3}{|\alpha_2|} \right) - \gamma_3(B_3) \left(A_1 - I_{\alpha_3,\beta_2} | \frac{A_2 - A_1}{|\alpha_3|} \right) \right]$$

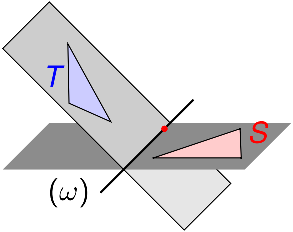
Other cases



We have to choose the origin in the intersection of the two planes.

More technical but OK.

Other cases



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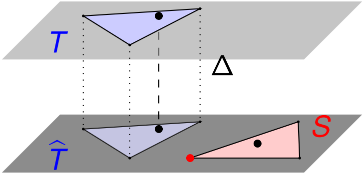

$\widehat{T}$ is the projection of T on the S plane and Δ becomes a parameter.

$$\|x - y\| = \left(\Delta^2 + \|x - \widehat{y}\|^2\right)^{\frac{1}{2}}$$

! We loose the homogeneity !

can appear with stratified media or thin plates

What happens

when we lose the homogeneity ?

If the triangles are in parallel planes

$$I = \int_{S\times T} \frac{dx\,dy}{\|x-y\|} = \int_{S\times\widehat{T}} \frac{dx d\widehat{y}}{\sqrt{\Delta^2 + \|x-\widehat{y}\|^2}}$$

Suppose

- $\Omega \subset \mathbb{R}^n$
- $f : (z,\Delta) \in \Omega \times \mathbb{R} \to \mathbb{R}$ is a homogeneous function of degre r according to z and Δ :

$$f(\lambda z, \lambda\Delta) = \lambda^r f(z,\Delta), \forall\lambda > 0$$

Then

$$\int_\Omega f(z,\Delta)dz = \int_{\partial\Omega} (z|\overrightarrow{\nu})\Delta^{r+n} \int_\Delta^{+\infty} \frac{f(z,u)}{u^{r+n+1}} du d\gamma_z$$

Proof.

By integration of the Euler's theorem on homogeneous functions and solving an O.D.E. □

What happens

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The second formula

$$\int_{\Omega} f(z,\Delta)dz = \int_{\partial\Omega} (z|\overrightarrow{\nu})\Delta^{r+n} \int_{\Delta}^{+\infty} \frac{f(z,u)}{u^{r+n+1}} du d\gamma_z$$

No reduction of the dimension !

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But if an explicit expression is found for the interior integral

$$\int_{\Omega} f(z,\Delta)dz = \int_{\partial\Omega} (z|\overrightarrow{\nu})F(z,\Delta)d\gamma_z$$

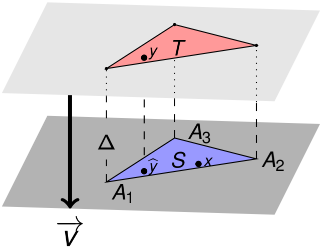
With,

$$F(z,\Delta) = \Delta^{r+n} \int_{\Delta}^{+\infty} \frac{f(z,u)}{u^{r+n+1}} du$$

Superposed Triangles

Double layer potential

$$J = \int_{S\times T} \frac{x-y}{\|x-y\|^3} dx\, dy$$
$$= \overrightarrow{v} \int_{S\times S} \frac{dx d\widehat{y}}{\left(\Delta^2 + \|x-\widehat{y}\|^2\right)^{3/2}}$$



Superposed Triangles

Double layer potential

$$
\begin{aligned}
J &= \int_{S\times T} \frac{x-y}{\|x-y\|^3}\, dx\, dy \\
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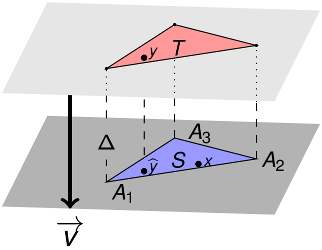
$$



After reductions

$$
J = 4\overrightarrow{v}\,|S| \sum_{k=1..3} \gamma_i \mathfrak{R}(A_k, \alpha_k, \Delta)
$$

Superposed Triangles

$$\Re(A_k, \alpha_k, \Delta) = \Bigg[s\,\frac{\sqrt{\Delta^2+\gamma_k^2+s^2}-\Delta}{2\gamma_k^2(\gamma_k^2+s^2)} - \frac{s \operatorname{arg\,sinh}\left(\frac{\sqrt{\gamma_k^2+s^2}}{\Delta}\right)}{\gamma_k^2\sqrt{\gamma_k^2+s^2}}$$
$$+\frac{1}{\gamma_k^2}\operatorname{arg\,sinh}\left(\frac{s}{\sqrt{\Delta^2+\gamma_k^2}}\right) + \frac{\gamma_k^2-\Delta^2}{2\Delta\gamma_k^3}\operatorname{arg\,tan}\left(\frac{s}{\gamma_k}\right) - \frac{(\gamma_k^2-\Delta^2)\,\pi\;\operatorname{sgn} s}{4\Delta\gamma_k^3}$$
$$+\frac{\gamma_k^2-\Delta^2}{2\Delta\gamma_k^3}\,\mathrm{Im}\left\{\operatorname{arg\,tanh}\left(\frac{\Delta^2+\gamma_k^2+i\gamma_k s}{\Delta\sqrt{\Delta^2+\gamma_k^2+s^2}}\right)\right\}\Bigg]_{s^- -\sigma}^{s^+ -\sigma}$$

Δ : the distance between the two planes
γ_k : the distance of A_k to α_k
s^+, s^- and σ : abscissas

Numerical Results

- Reduction implemented in Matlab (C++ in progress)

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Numerical Results

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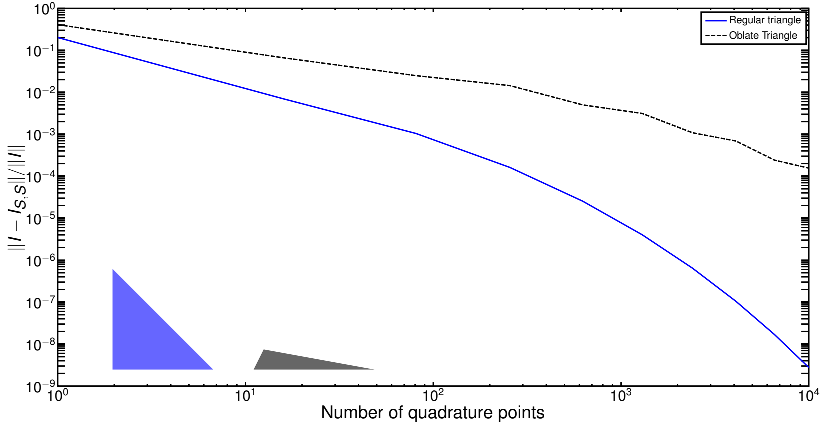
$$I = \int_{S \times S} \frac{dx\, dy}{\|x - y\|}$$

- We observe our result for the superposed triangles for the double layer potential when $\Delta \to 0$

$$J = \int_{S \times T} \frac{dx\, dy}{\|x - y\|^3}$$

Self-influence

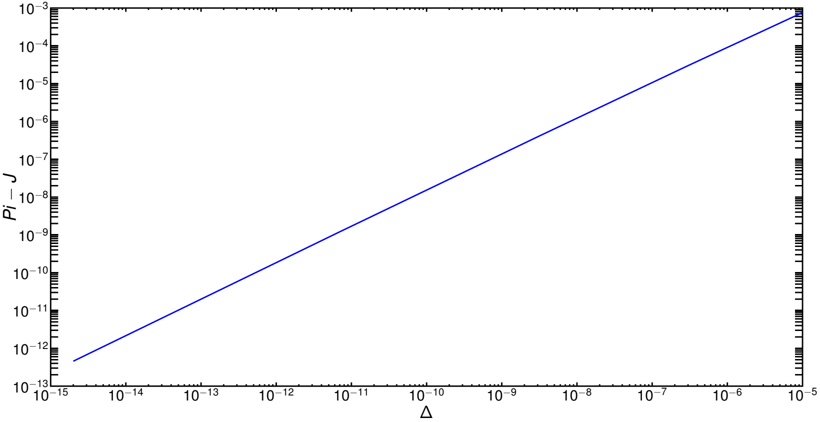
Comparison with Sauter & Schwab's Method



Sauter, S. and Schwab, C., *Boundary Element Methods*, Springer Series in Computational Mathematics, 2010

Superposed Triangles

The Solid Angle



Conclusion

The method

• Exact formulas for all S and T ; especially for singular and almost singular cases

• This technique relies on the homogeneity so we must expand the integrand in homogeneous terms

• This principle can be used for other integrands

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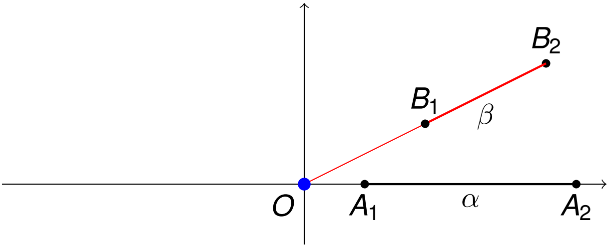
Work done & Perspectives

• Influence by constant density for single and double layers potentials is treated
• Linear basis function is in progress (self-influence done)
• With linear edge basis functions, Maxwell is partially treated
• Calculation for volumic potential is possible

Thank you for your attention !

The Almost Parallel Case



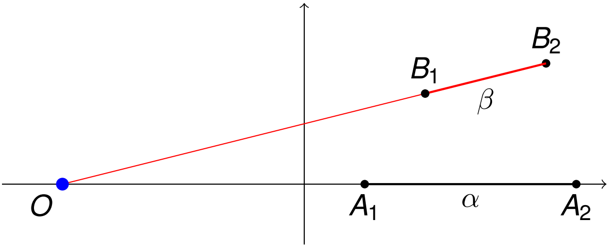
We obtain

$$I = \int_{\alpha\times\beta} \frac{dxdy}{\|x-y\|}$$
$$= s_1 R(A_1,\beta) - s_2 R(A_2,\beta) + t_1 R(B_1,\alpha) - t_2 R(B_2,\alpha)$$

with $R(a,\beta) = \int_\beta \frac{dx}{\|a-x\|}$

and $s_1 = \|O - A_1\|, t_1 = \|O - B_1\| \ldots$

The Almost Parallel Case



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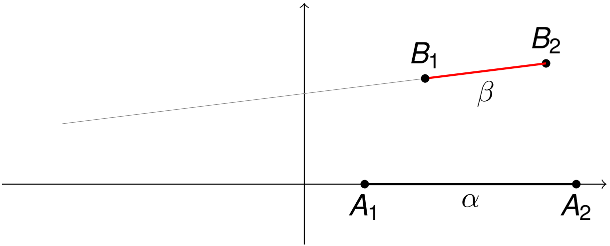
The Almost Parallel Case



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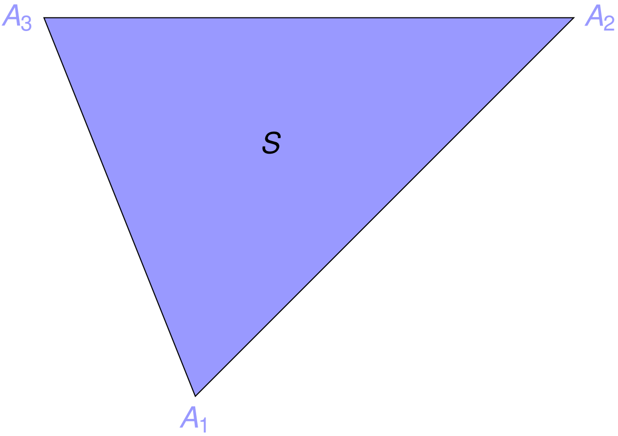
Almost parallel case



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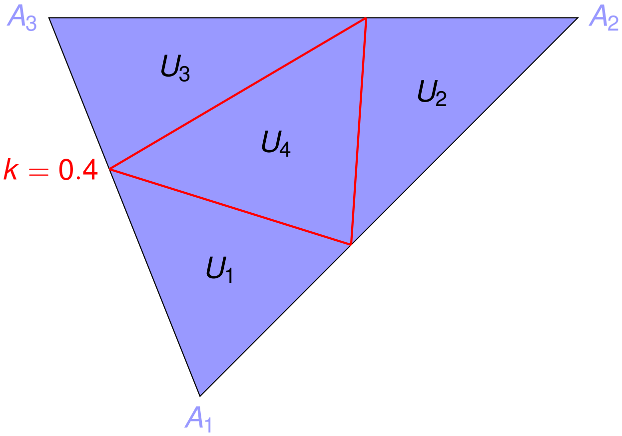
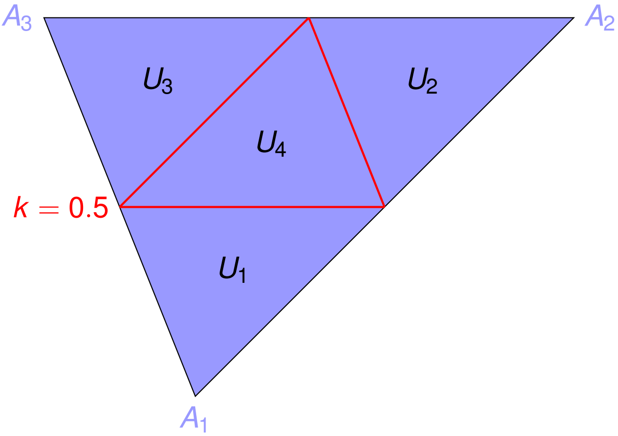
Almost parallel case



$$I = \int_{S\times S} \frac{dxdy}{\|x-y\|} = \sum_{i=1..4} \int_{U_i\times S} \frac{dxdy}{\|x-y\|}$$

Almost parallel case



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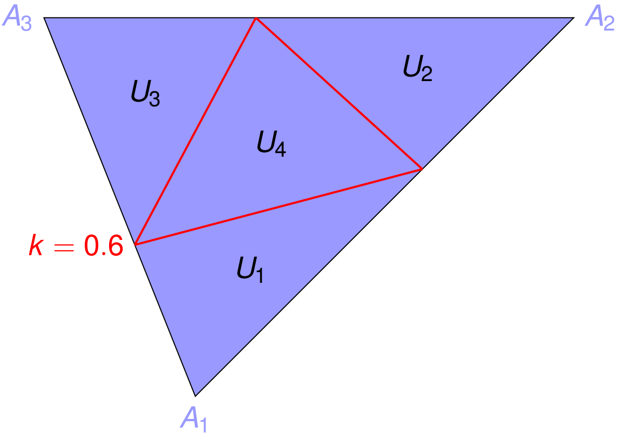
Almost parallel case



$$I = \int_{S \times S} \frac{dxdy}{\|x - y\|} = \sum_{i=1..4} \int_{U_i \times S} \frac{dxdy}{\|x - y\|}$$

Almost parallel case