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EXACT EVALUATION OF SINGULAR AND NEAR-SINGULAR INTEGRALS IN GALERKIN BEM

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Abstract. *We present a new method for the explicit evaluation of singular and near-singular integrals arising in Galerkin BEM. It is based on a recursive reduction of a m-dimensional integral into a linear combination of (m-1)-dimensional integrals. It leads to a linear combination of 1-dimensional regular integrals with factors depending only on geometric quantities. These integrals can be evaluated either numerically or explicitely. In this last case, a high degree of accuracy can be obtained, even in the case of near-singular integrals. This method relies upon the homogeneity of the singular part of the Green function, the regular part of which being subject to numerical cubature techniques. The method has appealing properties in terms of reliability, precision and flexibility.*

It is necessary to use plane polygons for the discretisation to ensure the homogeneity of the integrand, so curved triangles are not supported. Actually, we studied the case of plane triangles but any other plane polygones may be considered.

Our results include the cases of constant and linear basis functions, of Helmholtz and Maxwell equations, and of 2-D and 3-D geometries. In the present document, formulas for the reduction of the singular part of the single layer potential with piecewise constant basis functions are provided when triangles are coplanar, adjacent in secant planes and superposed in parallel planes. The reduction process with linear basis functions is presented and formulas for the cases of the self-influence and adjacent triangles are provided.

1 INTRODUCTION

The discretization of 3-D scattering problems by variational boundary element methods leads to the evaluation of such elementary integrals (see [1],[2] and [3]) as

$$\int_{S\times T} G(x,y)\; v(x)\, w(y)\; dx\, dy \text{ and } \int_{S\times T} \frac{\partial}{\partial n_y} G(x,y)\; v(x)\, w(y)\; dx\, dy, \qquad (1)$$

where v and w are polynomial basis functions, G is the Green kernel and S and T two planar polygons from the discretization of the boundary. Due to the singularity of the kernel, the numerical evaluation of these integrals may lead to inaccurate results when S and T are close to each other. We split G and its gradient into a regular part which involves classical numerical techniques and a singular part subject to our method. More specifically, the Green function of the 3-D Helmholtz equation writes as

$$G(x,y) = -\frac{1}{4\pi}\frac{1}{\|x-y\|} + H(\|x-y\|),$$

where H is an analytical function. The present method consists in recursively reducing the dimension of the integration domain of integrals with positively homogeneous integrands, using formulas (5) or (4) depending on the case, so as to obtain a linear combination of one-dimensional regular integrals, with coefficients depending on the relative positions of the triangles. Moreover explicit formulas can be derived for these integrals. We focus on the following integrals:

$$I(S,T) = \int_{S\times T} \frac{1}{\|x-y\|}\, dx\, dy \text{ and } I^{\phi_p,\phi_q}(S,T) = \int_{S\times T} \frac{\phi_p(x)\,\phi_q(y)}{\|x-y\|} dx\, dy. \qquad (2)$$

We compute $I(S,T)$ for coplanar triangles, then for adjacent triangles in secant planes, and finally, the case of superposed triangles is briefly presented. Whereafter, we discussed the calculation of $I^{\phi_p,\phi_q}(S,T)$ for identical or adjacent triangles. Other results for the singular part of the double layer for triangles in parallel planes, are available in an other article which is in review.

2 TOOLS AND NOTATIONS

We present some tools and notations needed for the calculations as the formulas used for the reduction and the notations used.

2.1 Integration of homogeneous functions

Let $f(x,d) \;: \Omega \subset \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ a positively homogeneous function of degree q. We denote $I(d) = \int_\Omega f(z,d)\, dz$ which by Euler's formula and Green's theorem satisfies the differential equation (see [4] and [5]) :

$$(q+n)\, I(d) = d I'(d) + \int_{\partial\Omega} (\overrightarrow{z}\,|\overrightarrow{\nu})\, f(z,d)\, d\gamma_z, \qquad (3)$$

where $\overrightarrow{\nu}$ is the exterior normal to Ω and $(\cdot|\cdot)$ is the scalar inner product.

Provided $d^{-(q+n)}\int_{\Omega} f(z,d)\,dz \to 0$ as $d \to +\infty$ one obtains

$$I(d) = d^{q+n}\int_{\partial\Omega} \left(\overrightarrow{z}\,|\overrightarrow{\nu}\right)\int_{d}^{+\infty} \frac{f(z,t)}{t^{q+n+1}}dt d\gamma_z. \quad (4)$$

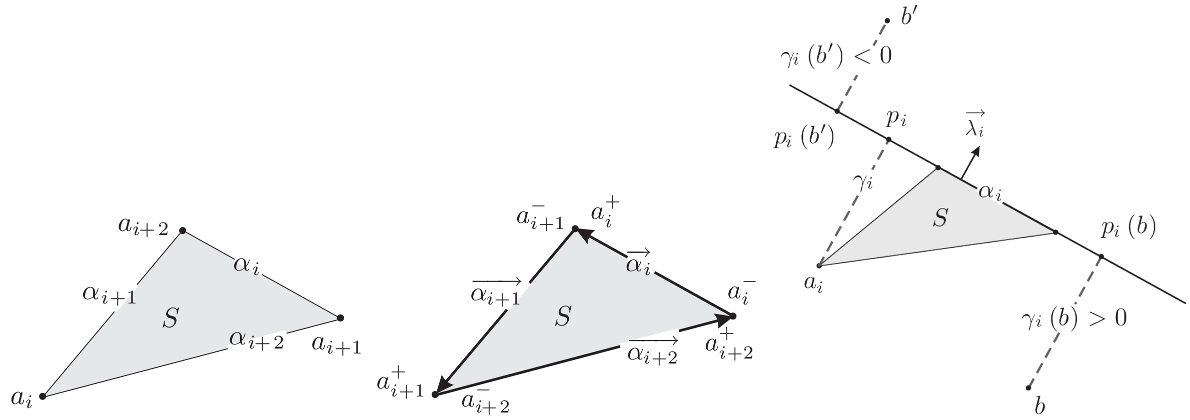
When $f(z,d)$ does not depend on d and $q+n \neq 0$ then

$$I = \frac{1}{q+n}\int_{\partial\Omega} \left(\overrightarrow{z}\,|\overrightarrow{\nu}\right) f(z)\, d\gamma_z. \quad (5)$$

As long as the inner integral in (4) can be explicitely evaluated, both formulas reduce an n-dimensional integral to an $(n-1)$ one. As Ω is an n-dimensional polyhedron (such as $S \times T$ with $n = 4$), $\left(\overrightarrow{z}\,|\overrightarrow{\nu}\right)$ is constant on each$(n-1)$-face of Ω, a simplification of crucial importance in the sequel. Formulas have been obtained for three types of geometrical configurations: S and T (i) coplanar, (ii) in secant planes and (iii) in parallel planes. All these cases are treated using formulas (5) or (4) or both, depending on the relative positions of triangles S and T.

2.2 Notations

The vertices of S are denoted by $a_i, i = 1,3$, those of T by $b_j, j = 1,3$, the opposite side to a_i by α_i and the opposite side to b_j by β_j. We use $|\alpha_i|$ the length of the side α_i, $\overrightarrow{\alpha_i}$ vector $a_{i+2} - a_{i+1} = a_i^+ - a_i^-$ and $\overrightarrow{\alpha_i}' = \overrightarrow{\alpha_i}/|\alpha_i|$, and similarly for T (see Figure 1(a)).



(a) General notations (b) Distances in S

Figure 1: Notations in S.

Projections Various kinds of projections and distances appear in our formulas : the projection of x on α is denoted by $p(x)$, the projection of y on β by $q(y)$ and one puts $p^{\pm} = p(b^{\pm}), q^{\pm} = q(a^{\pm})$. More specifically, p_i is the projection on α_i and q_j the projection on β_j, $\overrightarrow{\lambda_i}$ is the exterior normal to S along $\alpha_i, \gamma_i(x) = \left(p_i - x|\overrightarrow{\lambda_i}\right)$ the signed distance from x to α_i and $\delta_j(y)$ the signed distance from y to β_j. Finally we define $p_i = p_i(a_i)$, $q_j = q_j(b_j)$, $\gamma_i = \gamma_i(a_i) > 0$ and $\delta_j = \delta_j(b_j) > 0$ (see Figure 1(b)).

Abscissas On side α (resp β) the abscissa s (resp t) is defined with respect to an origin o_α (resp o_β) and a unitary direction vector $\overrightarrow{\alpha}' = \overrightarrow{\alpha}/|\alpha|$ (resp $\overrightarrow{\beta}' = \overrightarrow{\beta}/|\beta|$). The abscissas of the ends $a^\pm$ and $b^\pm$ are respectively denoted by $s^\pm$ and $t^\pm$, as well as the abscissas of $p^\pm$ and $q^\pm$ are denoted by $\sigma^\pm$ and $\tau^\pm$. The same notations with index i or j apply for a side α_i or β_j of S or T (see Figure 2).

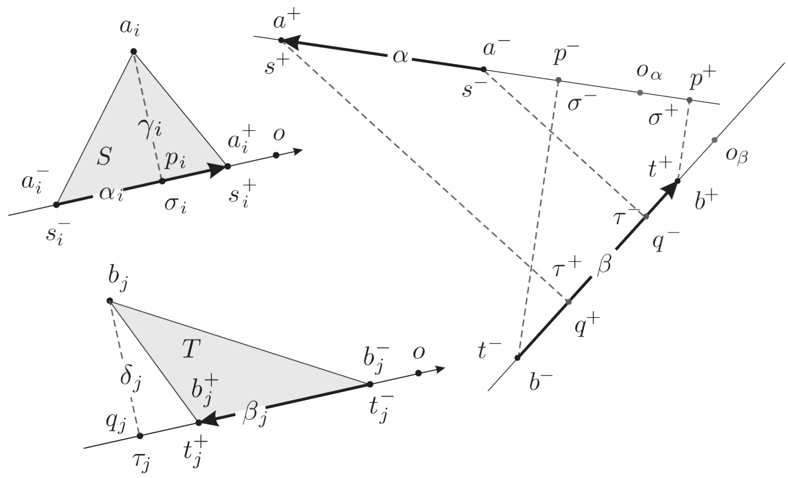


Figure 2: Abscissas

3 REDUCTION WITH CONSTANT BASIS FUNCTIONS

We start with the reduction of $I(S,T)$ in (2) when S and T are two arbitrary coplanar triangles, we detail the calculations and we finish with formulas for special cases : self-influence, adjacent triangles and triangles with a common vertex.

3.1 Arbitrary coplanar triangles

In the following, we use I in place of $I(S,T)$ from (2). We detail the entire reduction process to a linear combination of 1-D regular integrals with coefficients depending only on geometric quantities (signed distances, length of edges, area of triangles).

3.1.1 Reduction to dimension 3

Applying formula (5) to I with $\Omega = S \times T$ so $n = 4$, $z = (x,y)$ and $f(z) = \|x-y\|^{-1}$ from which $q = -1$, we obtain

$$I = \frac{1}{3}\int_{\partial(S\times T)} \frac{((x,y)|\overrightarrow{\nu})}{\|x-y\|}\partial(x,y). \tag{6}$$

As $\partial(S\times T) = \bigcup_{i=1..3}(\alpha_i \times T) \cup \bigcup_{j=1..3}(S \times \beta_j)$ and the scalar products can be written as $((x,y)|\overrightarrow{\nu})_{|\alpha_i\times S} = (x|\overrightarrow{\lambda}_i)_{|\alpha_i}$ and $((x,y)|\overrightarrow{\nu})_{|S\times\beta_j} = (y|\overrightarrow{\lambda}_j)_{|\beta_j}$, with a_i as origin, one has

$$\left(x|\overrightarrow{\lambda_{i+1}}\right) = \left(x|\overrightarrow{\lambda_{i+2}}\right) = 0,\ \left(x|\overrightarrow{\lambda_i}\right) = \gamma_i \text{ and } \left(y|\overrightarrow{\lambda_j}\right) = \delta_j(a_i),$$

as seen in Figure 3. Then, from formula (6):

$$I = \frac{\gamma_i}{3} U(\alpha_i, T) + \sum_{i=1}^{3} \frac{\delta_j(a_i)}{3} U(\beta_j, S), \tag{7}$$

$$\text{where } U(\alpha, T) = \int_{\alpha \times T} \frac{1}{\|x - y\|} d\gamma_x dy \tag{8}$$

is the influence coefficient between segment α and triangle T.

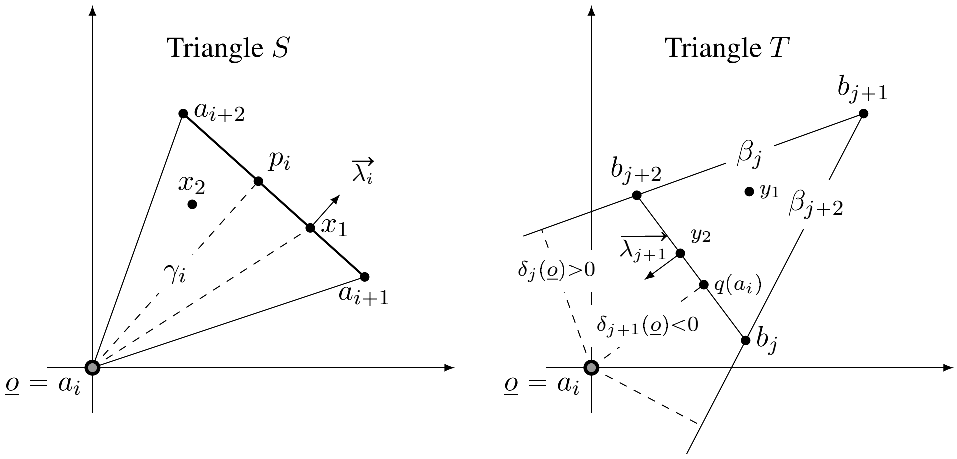


Figure 3: The boundary of $S \times T$: $(x_1, y_1) \in (\partial S \times T)$ and $(x_2, y_2) \in (S \times \partial T)$. We observe that according to the exterior normal on β_{j+1}, the signed distance $\delta_{j+1}(\underline{o})$ is negative.

3.1.2 Reduction to dimension 2

We choose a^- as origin, so that the function $\|x - y\|^{-1}$ be homogeneous on $\alpha \times T$. Now, we apply (5) with $n = 3$ and one obtains

$$U(\alpha, T) = \frac{1}{2} \int_{\partial(\alpha \times T)} \frac{((x, y)|\overrightarrow{\nu})}{\|x - y\|} \partial(x, y), \tag{9}$$

where $\overrightarrow{\nu}$ is the exterior normal to $\partial(\alpha \times T) = ((a^- \cup a^+) \times T) \bigcup (\alpha \times \partial T)$. From Figure 4, it is clear that

$$((x, y)|\overrightarrow{\nu})_{|a^- \times T} = 0,$$

as a consequence

$$U(\alpha, T) = \frac{|\alpha|}{2} P(a^+, T) + \sum_{j=1}^{3} \frac{\delta_j(a^-)}{2} Q(\alpha, \beta_j), \text{ where} \tag{10}$$

$$P(a, T) = \int_T \frac{1}{\|a - y\|} dy \text{ and } Q(\alpha, \beta) = \int_{\alpha \times \beta} \frac{1}{\|x - y\|} d\gamma_x d\gamma_y. \tag{11}$$

Actually $P(a, T)$ is the potential at a from T and $Q(\alpha, \beta)$ is the influence coefficient between α and β. Similarly,

$$U(\beta, S) = \frac{|\beta|}{2} P(b^+, S) + \sum_{i=1}^{3} \frac{\gamma_i(b^-)}{2} Q(\alpha_i, \beta). \tag{12}$$

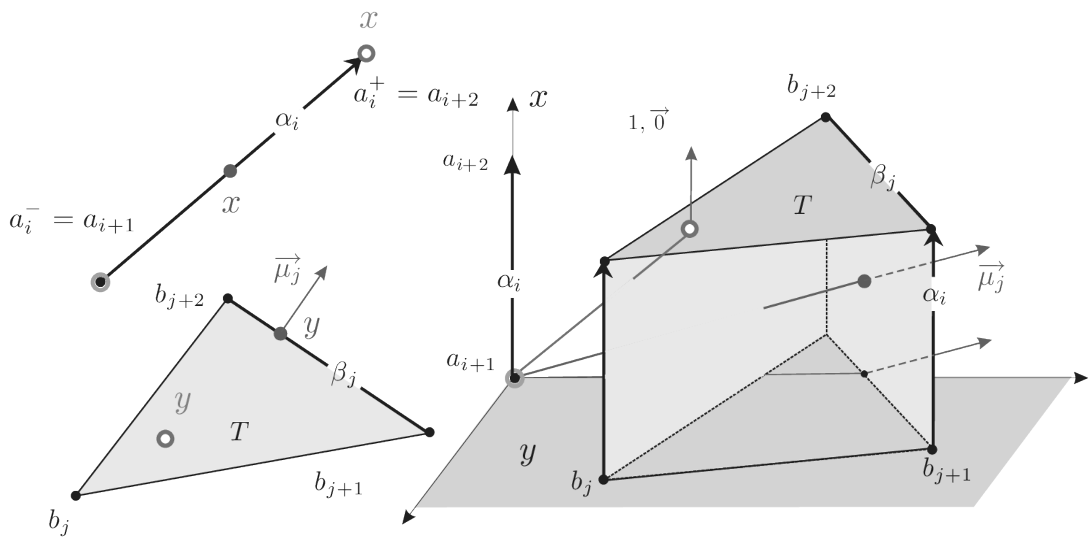


Figure 4: Boundary of $\alpha \times T$.

3.1.3 Reduction to dimension 1

We start the simplification with P. The reduction of Q needs to distinguish two cases: when the supports of the segments are secant and when they are parallel.

Calculation of $P(a, T)$ We only have the possibility of choosing a as the origin (see Figure 5), then using formula (5) we obtain

$$P(a,T) = \sum_{j=1}^{3} \delta_j(a) R(a, \beta_j), \text{ where} \tag{13}$$

$$R(a,\beta) = \int_\beta \frac{1}{\|a-y\|} d\gamma_y, \tag{14}$$

with R given by (70). As point a does not belong to segment β, the integrand in (14) is regular.

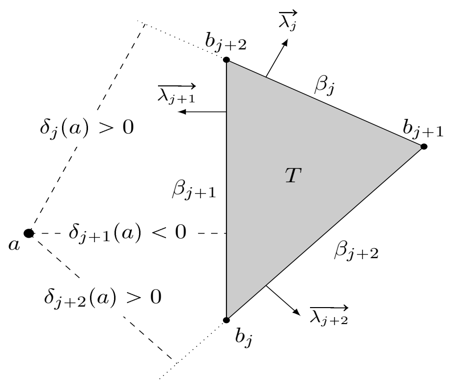
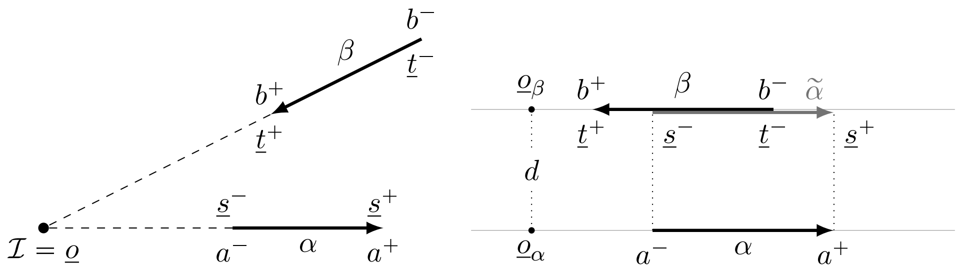


Figure 5: Calculation of $P(a, T)$

Calculation of $Q(\alpha, \beta)$**: the case of secant supports** When the supports of α and β are secant, we choose the intersection $\mathcal{I}$ as the new origin $\underline{o}$. The respective abscissas of $a^\pm$ and $b^\pm$



(a) Calculation of Q with secant supports (b) Calculation of Q: parallel segments

Figure 6: Calculation of Q : segments with secants supports and parallel supports

with respect to this origin are denoted by $\underline{s}^{\pm}$ and $\underline{t}^{\pm}$ (see Figure 6(a)), then one has

$$Q(\alpha,\beta) = \sum_{k=\pm} k\, \underline{s}^k\, R(a^k,\beta) + \sum_{l=\pm} l\, \underline{t}^l\, R(b^l,\alpha). \tag{15}$$

Calculation of $Q(\alpha,\beta)$: the case of parallel supports When the supports of α and β are parallel, no more origin makes the integrand homogeneous with respect to the integration variable. We are thus led to use formula (4) instead of (5). Accordingly, we denote by d the distance between α and β, by $\tilde{\alpha}$ the projection of α on the support of β, and the ends $\underline{o}_\alpha$ and $\underline{o}_\beta$ are the new origins on α and β (see Figure 6(b)). Applying formula (4), we have

$$Q(\alpha,\beta) = \int_{\tilde{\alpha}\times} \frac{1}{\sqrt{d^2 + \|\tilde{x} - y\|^2}} d\gamma_{\tilde{x}} d\gamma_y \tag{16}$$

$$= d \int_{\partial(\tilde{\alpha}\times\beta)} ((x,y)|\overrightarrow{\nu}) \int_d^{+\infty} \frac{du}{u^2\sqrt{u^2 + \|\tilde{x} - y\|^2}} \partial(\tilde{x}, y). \tag{17}$$

The corresponding formula to (15) is thus

$$Q(\alpha,\beta) = \sum_{k=\pm} k\, \underline{s}^k\, \mathcal{R}(\tilde{a}^k,\beta,d) + \sum_{l=\pm} l\, \underline{t}^l\, \mathcal{R}(b^l,\tilde{\alpha},d) \text{ with} \tag{18}$$

$$\mathcal{R}(\tilde{a},\beta,d) = \int_\beta f_1(d, \|\tilde{a} - y\|) d\gamma_y, \text{ and} \tag{19}$$

$$\mathrm{f}_1(d,\eta) = d^1 \int_d^{+\infty} \frac{du}{u^2\sqrt{u^2+\eta^2}} = \frac{\sqrt{d^2+\eta^2} - d}{\eta^2}, \tag{20}$$

with $\mathcal{R}$ given by (71).

The almost-parallel case When the intersection $\underline{o}$ of the supports of α and β is excessively remote, then $\underline{s}^k$ and $\underline{t}^l$ in formula (15) take very large values as compared to $|\alpha|$ and $|\beta|$, resulting in an indeterminate form and a cancellation of significant digits. The strategy we adopt consists in replacing α and β by two close segments which are actually parallel once $|\underline{s}^k|$ and $|\underline{t}^l|$ exceed some prescribed value.

3.2 Special configurations of coplanar triangles

When the intersection of S and T is not empty (self-influence, triangles with a common edge or vertex), using common vertices as origin, we obtain simpler expressions (see Figure 7).

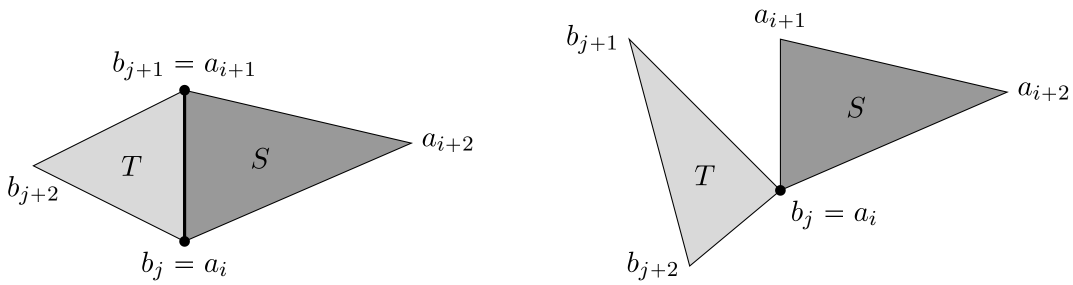


Figure 7: Coplanar adjacent triangles and coplanar triangles with a common vertex

Self-influence When S and T are identical, parallelism between segments does not occur, which leads to this symmetric expression:

$$I = \frac{2}{3}|S| \sum_{i=1}^{3} \gamma_i R(a_i, \alpha_i). \tag{21}$$

Adjacent coplanar triangles We use formulas (15) or (18) to calculate Q, depending on the fact there is parallelism or not, thus

$$I = \frac{|S|}{3} P(a_{i+2}, T) + \frac{|T|}{3} P(b_{j+2}, S) + \frac{\gamma_i \delta_{j+1}}{6} Q(\alpha_i, \beta_{j+1}) + \frac{\gamma_{i+1} \delta_j}{6} Q(\alpha_{i+1}, \beta_j), \tag{22}$$

with P and Q both defined by (11).

Coplanar triangles with a common vertex We choose the common vertex as origin, then

$$I = \frac{\gamma_i}{3} U(\alpha_i, T) + \frac{\delta_j}{3} U(\beta_j, S), \tag{23}$$

with U defined by (10).

3.3 Adjacent triangles in secant planes

Up to dimension 2, the calculation of $I(S,T)$ when S and T are two adjacent triangles in secant planes is similar to the coplanar case (see formula (22)). The calculations of P and Q differ.

Calculation of P The evaluation of $P(a,T)$ requires the projection of a on the plane of T (see Figure 8). We denote by $\widehat{a}$ the projection and we introduce the distance $h = \|a - \widehat{a}\|$, then

$$P(a,T) = \int_T \frac{dx}{\sqrt{h^2 + \|x - \widehat{a}\|^2}}, \tag{24}$$

which is similar to (18). By (13), we obtain the final expression

$$P(a,T) = \sum_{j=1}^{3} \delta_j(\widehat{a}) \mathcal{R}\left(\widehat{a}, \beta_j, h\right) \text{ and } P(b,S) = \sum_{i=1}^{3} \gamma_i(\widehat{b}) \mathcal{R}\left(\widehat{b}, \alpha_i, h\right), \tag{25}$$

with $\mathcal{R}$ defined by (71).

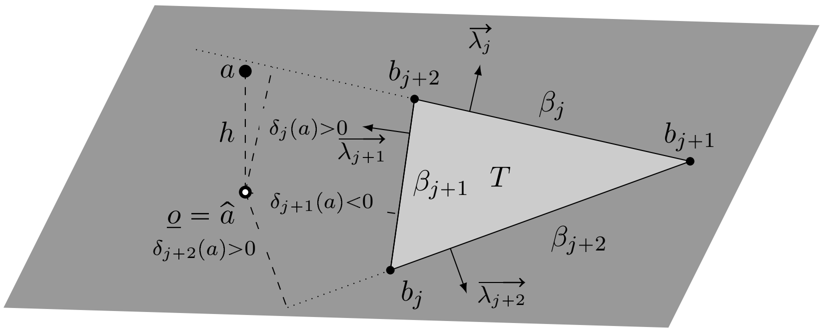


Figure 8: Calculation of $P(a, T)$ when a does not belong to the plane of T.

Calculation of Q When triangles are adjacent, α and β can not be parallel (see Figure 9). The determination of the common perpendicular to α and β is required. We define $\breve{\alpha}$ and $\breve{a}$ the projections of α and a on the parallel plane to α which contains β and $h = \|a^+ - \breve{a}^+\|$. One has

$$Q(\alpha, \beta) = \int_{\breve{\alpha}\times\beta} \frac{d\breve{x}dy}{\sqrt{h^2 + \|\breve{x} - y\|^2}}, \tag{26}$$

which looks like (16). With the origin $\underline{o}$ chosen as the intersection of the supports of $\breve{\alpha}$ and β, we obtain

$$Q(\alpha, \beta) = \sum_{k=\pm} k\, \underline{\breve{s}}^k \mathcal{R}\left(\breve{a}^k, \beta, h\right) + \sum_{l=\pm} l\, \underline{t}^l \mathcal{R}\left(b^l, \breve{\alpha}, h\right). \tag{27}$$

It is worth pointing out that two arbitrary triangles in secant planes may have parallel edges.

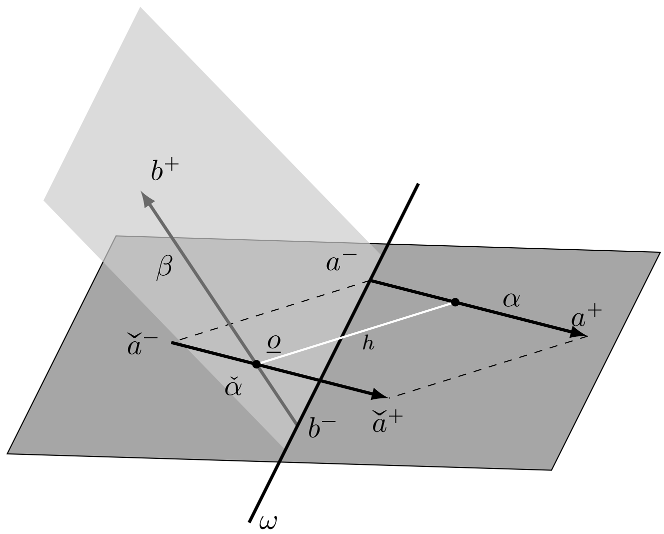


Figure 9: Segments in secant planes with the common perpendicular to α and β in white and (ω) denote the intersection of the two planes.

3.4 Superposed triangles

The method is valid for all the geometric configurations. We present the case of superposed triangles. In this case, formula (4) is used from the first step of the reduction process, so we obtain more complicated expressions (see expression (29)). Let T be a triangle such as its projection $\widehat{T}$ on the plane of S is exactly S (see Figure 10). After three successive reductions using formula (4), one obtains the following expression, which is similar to (21)

$$I(S,T) = \frac{2\,|S|}{3} \sum_{i=1}^{3} \gamma_i \mathfrak{R}(a_i, \alpha_i, \Delta), \text{ with} \tag{28}$$

$$\begin{aligned}\mathfrak{R}(a,\alpha,\Delta) = \Bigg[& \frac{3s\Delta^2}{\sqrt{d^2+s^2}d^2} \arg\sinh\left(\frac{\sqrt{d^2+s^2}}{\Delta}\right) + \frac{(d^2-3\Delta^2)}{d^2} \arg\sinh\left(\frac{s}{\sqrt{\Delta^2+d^2}}\right) \\ & - \frac{s\Delta^2}{d^2} \frac{\sqrt{\Delta^2+d^2+s^2}-\Delta}{d^2+s^2} - \frac{\Delta(3d^2-\Delta^2)}{d^3} \arg\tan\left(\frac{s}{d}\right) \\ & - \frac{\Delta(3d^2-\Delta^2)}{d^3} \mathrm{Im}\left\{ \arg\tanh\left(\frac{\Delta^2+d^2+isd}{\Delta\sqrt{\Delta^2+d^2+s^2}}\right)) - \frac{i\pi}{2}\,\mathrm{sgn}(s) \right\} \Bigg]_{s^+}^{s^-}.\end{aligned} \tag{29}$$

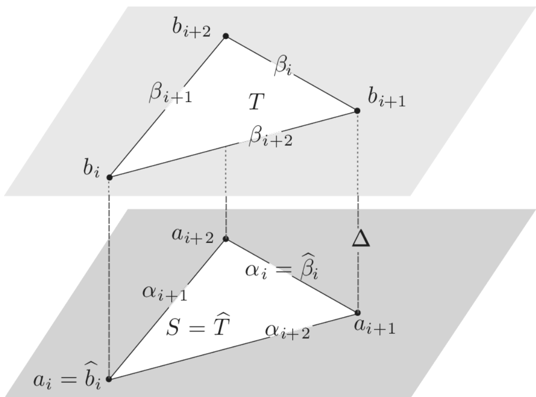


Figure 10: Superposed triangles

4 REDUCTION WITH LINEAR BASIS FUNCTIONS

This section is devoted to the calculation of I^{ϕ_p,ϕ_q} defined by (2). We begin with the self-influence case, then we provide formulas for the case of adjacent triangles.

Basis functions Let ϕ_p a basis function, such as $\phi_p(x) = 1-(x-a_p|\overrightarrow{e_p})$, with some properties: $\phi_p(a_p) = 1$ and $\phi_p(x)_{|\alpha_p} = 0$. The preservation of the homogeneity will lead an extensive use of formula (30):

$$\phi_p(x) = \phi_p(\underline{o}) + (\underline{x}|\overrightarrow{e_p}), \tag{30}$$

where $\underline{o}$ is the new origin and $\underline{x}$ the coordinates of x according to $\underline{o}$.

4.1 Self-influence coefficient with linear basis functions

We are interested in the calculation of

$$I^{p,q} = \int_{S\times S} \frac{\phi_p(x)\,\phi_q(y)}{\|x-y\|} dx\,dy \qquad \text{for } p = 1,3 \text{ and } q = 1,3. \tag{31}$$

The diagonal coefficient is detailed; the other coefficients are given without any details. Due to the repeat use of formula (30), the amount of calculations is greater than for the case of constant basis functions.

4.1.1 Diagonal coefficients

The expression of the diagonal coefficients is

$$I^{i,i} = \int_{S\times S} \frac{\phi_i(x)\,\phi_i(x)}{\|x-y\|} dx\,dy, \tag{32}$$

with choosing a_i as origin. Since $\phi_i(a_i) = 1$ we have $\phi_i(x) = 1 + (\underline{x}|\overrightarrow{e_i})$, consequently

$$I^{i,i} = \int_{\underline{S}\times\underline{S}} \frac{d\underline{x}d\underline{y}}{\|\underline{x}-\underline{y}\|} + 2\int_{\underline{S}\times\underline{S}} \frac{(\underline{x}|\overrightarrow{e_i})}{\|\underline{x}-\underline{y}\|} d\underline{x}d\underline{y} + \int_{\underline{S}\times\underline{S}} \frac{(\underline{x}|\overrightarrow{e_i})\,(\underline{y}|\overrightarrow{e_i})}{\|\underline{x}-\underline{y}\|} d\underline{x}d\underline{y}. \tag{33}$$

From (5), we obtain

$$\begin{aligned} I^{i,i} = {} & \frac{2\gamma_i}{3}\int_{\underline{\alpha_i}\times\underline{S}} \frac{d\gamma_{\underline{x}}d\underline{y}}{\|\underline{x}-\underline{y}\|} + \frac{2\gamma_i}{4}\int_{\underline{\alpha_i}\times\underline{S}} \frac{(\underline{x}|\overrightarrow{e_i})}{\|\underline{x}-\underline{y}\|} d\gamma_{\underline{x}}d\underline{y} + \frac{\gamma_i}{2}\int_{\underline{S}\times\underline{\alpha_i}} \frac{(\underline{x}|\overrightarrow{e_i})}{\|\underline{x}-\underline{y}\|} d\underline{x}d\gamma_{\underline{y}} \\ & + \frac{\gamma_i}{5}\int_{\underline{\alpha_i}\times\underline{S}} \frac{(\underline{x}|\overrightarrow{e_i})(\underline{y}|\overrightarrow{e_i})}{\|\underline{x}-\underline{y}\|} d\gamma_{\underline{x}}d\underline{y} + \frac{\gamma_i}{5}\int_{\underline{S}\times\underline{\alpha_i}} \frac{(\underline{x}|\overrightarrow{e_i})(\underline{y}|\overrightarrow{e_i})}{\|\underline{x}-\underline{y}\|} d\underline{x}d\gamma_{\underline{y}}, \end{aligned} \tag{34}$$

then going back to the initial origin and using that $\phi_i(x)_{|\alpha_i} = 0$, one obtains

$$\begin{aligned} I^{i,i} &= \frac{2\gamma_i}{30}U(\alpha_i,S) + \frac{\gamma_i}{20}\left\{\int_{\alpha_i\times S} \frac{\phi_i(y)}{\|x-y\|} d\gamma_x\,dy + \int_{S\times\alpha_i} \frac{\phi_i(x)}{\|x-y\|} dx\,d\gamma_y\right\} \\ &= \frac{\gamma_i}{15}U(\alpha_i,S) + \frac{\gamma_i}{10}U^{\phi_i}(\alpha_i,S), \end{aligned} \tag{35}$$

with

$$U^{\phi}(\alpha,T) = \int_{\alpha\times T} \frac{\phi(y)}{\|x-y\|} d\gamma_x\,dy. \tag{36}$$

We choose $\underline{o} = a_{i+1}$ as origin, then

$$U^{\phi_i}(\alpha_i,S) = \frac{|\alpha_i|}{3}P^{\phi_i}(a_{i+2},S) + \frac{\gamma_{i+1}}{3}Q^{1,\phi_i}(\alpha_i,\alpha_{i+1}), \tag{37}$$

where

$$P^{\phi}(a,T) = \int_T \frac{\phi(x)}{\|x-a\|} dx \text{ and } Q^{v,w}(\alpha,\beta) = \int_{\alpha\times\beta} \frac{v(x)\,w(y)}{\|x-y\|} d\gamma_x\,dy\gamma_y, \tag{38}$$

with v or w replaced by 1 in the case of one basis function. The last step provides

$$U^{\phi_i}(\alpha_i,S) = \frac{|\alpha_i|\gamma_{i+2}}{6}R^{\phi_i}(a_{i+2},\alpha_{i+2}) + \frac{\gamma_{i+1}|\alpha_i|}{6}R^{\phi_i}(a_{i+1},\alpha_{i+1}) + \frac{\gamma_{i+1}|\alpha_{i+1}|}{6}R(a_i,\alpha_i), \tag{39}$$

where,

$$R^{\phi_i}(b,\alpha) = \int_\alpha \frac{\phi_i(x)}{\|x-b\|} d\gamma_x, \tag{40}$$

see (75). Finally, from (10) and (39), we obtain the final result:

$$\begin{aligned} I^{i,i} = {} & \frac{|S|}{30}\left(\gamma_{i+2}R^{\phi_i}(a_{i+2},\alpha_{i+2}) + \gamma_{i+1}R^{\phi_i}(a_{i+1},\alpha_{i+1})\right) \\ & + \frac{|S|\gamma_i}{30}R(a_i,\alpha_i) + \sum_{i=1}^{3}\frac{|S|\gamma_i}{15}R(a_i,\alpha_i). \end{aligned} \tag{41}$$

4.1.2 Others coefficients

For the two other coefficients the reduction process is similar:

$$\begin{aligned}I^{i,l} &= \sum_{i=1}^{3} \frac{|S|\gamma_i}{30} R(a_i,\alpha_i) + \frac{|S|}{60}\left[\gamma_l R(a_l,\alpha_l) + \gamma_i R(a_i,\alpha_i)\right] \\ &+ \frac{|S|}{60}\left[\gamma_i R^{\phi_l}(a_i,\alpha_i) + \gamma_m R^{\phi_l}(a_m,\alpha_m) + \gamma_{i+1} R^{\phi_i}(a_{i+1},\alpha_{i+1}) + \gamma_{i+2} R^{\phi_i}(a_{i+2},\alpha_{i+2})\right] \\ &+ \frac{|S|}{30}\left[\gamma_i R^{\phi_l}(a_i,\alpha_i) + \gamma_l R^{\phi_i}(a_l,\alpha_l)\right],\end{aligned} \tag{42}$$

with $(l,m) = (i+1,i+2)$ or $(i+2,i+1)$. Formulas (41) and (42) provide all terms of the elementary matrix.

4.2 Adjacent triangles with linear basis functions

Up to dimension 2, the process is identical for coplanar triangles or for ones in secant planes.

4.2.1 Matrix coefficients

Using symmetry, the following formulas are sufficient to calculate all the coefficients of the elementary matrix. For the first diagonal coefficient, we start the reduction process choosing $\underline{o} = a_i = b_j$ (see Figure 7), thus

$$\begin{aligned}I^{i,j} &= \frac{|S|}{30} P(a_{i+2},T) + \frac{|T|}{30} P(b_{j+2},S) + \frac{\gamma_i\,\delta_{j+1}}{60} Q(\alpha_i,\beta_{j+1}) + \frac{\delta_j\,\gamma_{i+1}}{60} Q(\alpha_{i+1},\delta_j) \\ &+ \frac{|S|}{30} P^{\phi_j}(a_{i+2},T) + \frac{|T|}{30} P^{\phi_i}(b_{j+2},S) \\ &+ \frac{\gamma_i\,\delta_{j+1}}{60} Q^{1,\phi_j}(\alpha_i,\beta_{j+1}) + \frac{\gamma_{i+1}\,\delta_j}{60} Q^{\phi_i,1}(\alpha_{i+1},\beta_j).\end{aligned} \tag{43}$$

By symmetry we obtain the coefficient $I^{i+1,j+1}$: the first line of (43) is unchanged, on the second and the third lines, we have to replace the basis function in P^{ϕ} and $Q^{v,w}$ and on the third line, $Q^{v,w}(\alpha_i,\beta_{j+1})$ becomes $Q^{v,w}(\alpha_{i+1},\beta_j)$ with the corresponding coefficients; thereby

$$\begin{aligned}I^{i+1,j+1} &= \frac{|S|}{30} P(a_{i+2},T) + \frac{|T|}{30} P(b_{j+2},S) + \frac{\gamma_i\,\delta_{j+1}}{60} Q(\alpha_i,\beta_{j+1}) + \frac{\delta_j\,\gamma_{i+1}}{60} Q(\alpha_{i+1},\delta_j) \\ &+ \frac{|S|}{30} P^{\phi_{j+1}}(a_{i+2},T) + \frac{|T|}{30} P^{\phi_{i+1}}(b_{j+2},S) \\ &+ \frac{\gamma_{i+1}\,\delta_j}{60} Q^{1,\phi_{j+1}}(\alpha_{i+1},\beta_j) + \frac{\gamma_i\,\delta_{j+1}}{60} Q^{\phi_{i+1},1}(\alpha_i,\beta_{j+1}).\end{aligned} \tag{44}$$

The three following coefficients

$$\begin{aligned}I^{i,j+1} &= \frac{|S|}{60}\left(P(a_{i+2},T) + 2P^{\phi_{j+1}}(a_{i+2},T)\right) + \frac{|T|}{60}\left(P(b_{j+2},S) + 2P^{\phi_i}(b_{j+1},S)\right) \\ &+ \frac{\gamma_{i+1}\delta_j}{120}\left(Q(\alpha_{i+1},\beta_j) + 2Q^{1,\phi_{j+1}}(\alpha_{i+1},\beta_j) + 2Q^{\phi_i,1}(\alpha_{i+1},\beta_j)\right) \\ &+ \frac{\gamma_{i+1}\delta_j}{20} Q^{\phi_i,\phi_{j+1}}(\alpha_{i+1},\beta_j) + \frac{\gamma_i\delta_{j+1}}{120} Q(\alpha_i,\beta_{j+1}),\end{aligned} \tag{45}$$

$$\begin{aligned} I^{i,j+2} = & \frac{|T|}{10} P^{\phi_i}(b_{j+2}, S) + \frac{|T|}{30} P(b_{j+2}, S) + \frac{|S|}{30} P^{\phi_{j+2}}(a_{i+2}, T) \\ & + \frac{\gamma_{i+1}\delta_j}{60} Q^{1,\phi_{j+2}}(\alpha_{i+1}, \beta_j) + \frac{\gamma_i \delta_{j+1}}{60} Q^{1,\phi_{j+2}}(\alpha_i, \beta_{j+1}) \\ & + \frac{\gamma_{i+1}\delta_j}{20} Q^{\phi_i,\phi_{j+2}}(\alpha_{i+1}, \beta_j), \end{aligned} \tag{46}$$

and

$$\begin{aligned} I^{i+1,j+2} = & \frac{|T|}{10} P^{\phi_{i+1}}(b_{j+2}, S) + \frac{|T|}{30} P(b_{j+2}, S) + \frac{|S|}{30} P^{\phi_{j+2}}(a_{i+2}, T) \\ & + \frac{\gamma_{i+1}\delta_j}{60} Q^{1,\phi_{j+2}}(\alpha_{i+1}, \beta_j) + \frac{\gamma_i \delta_{j+1}}{60} Q^{1,\phi_{j+2}}(\alpha_i, \beta_{j+1}) \\ & + \frac{\gamma_i \delta_{j+1}}{20} Q^{\phi_{i+1},\phi_{j+2}}(\alpha_{i+1}, \beta_j), \end{aligned} \tag{47}$$

provide respectively $I^{i+1,j}$, $I^{i+2,j}$ and $I^{i+2,j+1}$ by inversion of S and T. Finally,

$$\begin{aligned} I^{i+2,j+2} = & \frac{|S|}{10} P^{\phi_{j+2}}(a_{i+2}, T) + \frac{|T|}{10} P^{\phi_{i+2}}(b_{j+2}, S) \\ & + \frac{\gamma_i \delta_{j+1}}{20} Q^{\phi_{i+2},\phi_{j+2}}(\alpha_i, \beta_{j+1}) + \frac{\gamma_{i+1}\delta_j}{20} Q^{\phi_{i+2},\phi_{j+2}}(\alpha_{i+1}, \beta_j). \end{aligned} \tag{48}$$

Now, we detail the calculations of P^v, $Q^{1,w}$ and $Q^{v,w}$.

4.2.2 The case of triangles in secant planes

Calculation of $P^{\phi_q}(a,T)$ With $\widehat{a}$ as origin (see Figure 8), the reduction is similar to (24):

$$\begin{aligned} P^{\phi_q}(a,T) = & \int_T \frac{\phi_q(y)}{\sqrt{h^2 + \|y - \widehat{a}\|^2}} dy = \delta_{q+1}(\widehat{a}) \mathcal{S}^{\phi_q}(\widehat{a}, \beta_{q+1}, h) \\ & + \delta_{q+2}(\widehat{a}) \mathcal{S}^{\phi_q}(\widehat{a}, \beta_{q+2}, h) + \phi_q(\widehat{a}) \sum_{l=1}^{3} \delta_l(\widehat{a}) \mathcal{T}(\widehat{a}, \beta_l, h), \end{aligned} \tag{49}$$

with

$$\mathcal{S}^{\phi}(\widehat{a}, \beta, h) = \int_\beta \phi(y) f_2(h, \|\widehat{a} - y\|) d\gamma_y, \tag{50}$$

$$\text{and } \mathcal{T}(\widehat{a}, \beta, h) = \int_\beta f_1(h, \|y - \widehat{a}\|) - f_2(h, \|y - \widehat{a}\|) d\gamma_y, \tag{51}$$

where f_1 and f_2 are defined by

$$f_c(h, \eta) = h^c \int_h^{+\infty} \frac{du}{u^{c+1}\sqrt{u^2 + \eta^2}}. \tag{52}$$

The expression of f_1 is already given by (20) and

$$f_2(h, \eta) = \frac{\sqrt{h^2 + \eta^2}}{2\eta^2} - \frac{h^2}{2\eta^3} \arg\sinh\left(\frac{\eta}{h}\right). \tag{53}$$

Calculation of $Q^{1,\phi_q}(\alpha,\beta)$ We use a same approach as in (27). The origin $\underline{o}$ is chosen as the intersection of the supports of $\breve{\alpha}$ and β (see Figure 9), so

$$\begin{aligned} Q^{1,\phi_q}(\alpha,\beta) &= \int_{\breve{\alpha}\times\beta} \frac{\phi_q(y)}{\sqrt{h^2+\|\breve{x}-y\|^2}} d\gamma_{\breve{x}}\, d\gamma_y \\ &= \phi_q(\underline{o}) \left\{ \sum_{k=\pm} k\, \breve{s}^k \mathcal{T}(\breve{a}^k,\beta,h) + \sum_{l=\pm} l\, t^l \mathcal{T}(b^l,\breve{\alpha},h) \right\} \\ &+ \sum_{k=\pm} k\, \breve{s}^k \mathcal{S}^{\phi_j}(\breve{a}^k,\beta,h) + \sum_{l=\pm} l\, t^l \phi_j(b^l) \mathcal{S}(b^l,\breve{\alpha},h), \end{aligned} \tag{54}$$

$$\text{with } \mathcal{S}(b,\breve{\alpha},h) = \mathcal{S}^1(b,\breve{\alpha},h) = \int_{\breve{\alpha}} f_2(h,\|\breve{x}-b\|) d\gamma_{\breve{x}}, \tag{55}$$

and $\underline{\breve{s}}^k$ is the abscissa of $\breve{a}^k$ according to the origin $\underline{o}$. $\mathcal{S}, \mathcal{S}^v$ and $\mathcal{T}$ are respectively given by (76), (81) and (83).

Calculation of $Q^{\phi_p,\phi_q}(\alpha,\beta)$ We obtain these new expressions of the two basis functions:

$$\phi_p(x) = 1 + \mu \frac{(\breve{x}-\breve{a}_p|\overrightarrow{\alpha})}{|\alpha|^2} = C_\alpha + \mu \frac{(\breve{x}-\mathcal{I}|\overrightarrow{\alpha})}{|\alpha|^2} \tag{56}$$

$$\phi_q(y) = 1 + \nu \frac{(y-b_q|\overrightarrow{\beta})}{|\beta|^2} = C_\beta + \nu \frac{(y-\mathcal{I}|\overrightarrow{\beta})}{|\beta|^2}, \tag{57}$$

$$\text{with } \mu = 1 \text{ if } \breve{a}_p = \breve{a}^+, -1 \text{ if } \breve{a}_p = \breve{a}^- ;\ \nu = 1 \text{ if } b_q = b^+, -1 \text{ if } b_q = b^-,$$

and $C_\alpha = 1 + \frac{\mu(\mathcal{I}-\breve{a}_p|\overrightarrow{\alpha})}{|\alpha|^2}$ and $C_\beta = 1 + \frac{\nu(\mathcal{I}-b_q|\overrightarrow{\beta})}{|\beta|^2}$, where $\mathcal{I}$ is the intersection of the supports of $\breve{\alpha}$ and β. Using formula (4) with $\mathcal{I}$ as origin, one obtains

$$\begin{aligned} Q^{\phi_p,\phi_q}(\alpha,\beta) &= C_\alpha C_\beta \Big\{ \sum_{k=\pm} k\, \breve{s}^k \mathcal{R}(\breve{a}^k,\beta,h) + \sum_{l=\pm} l\, t^l \mathcal{R}(b^l,\breve{\alpha},h) \Big\} \\ &+ \sum_{k=\pm} k\, \breve{s}^k \Big\{ \frac{C_\alpha \nu}{|\beta|^2} \widetilde{\mathcal{V}}(\breve{a}^k,\mathcal{I},\beta,h) + \frac{C_\beta \mu}{|\alpha|^2} (\breve{a}^k-\mathcal{I}|\overrightarrow{\alpha}) \mathcal{S}(b^l,\breve{\alpha},h) \Big\} \\ &+ \sum_{l=\pm} l\, t^l \Big\{ \frac{C_\alpha \nu}{|\beta|^2} (b^l-\mathcal{I}|\overrightarrow{\beta}) \mathcal{S}(b^l,\breve{\alpha},h) + \frac{C_\beta \mu}{|\alpha|^2} \widetilde{\mathcal{V}}(b^l,\mathcal{I},\breve{\alpha},h) \Big\} \\ &+ \frac{\mu\,\nu}{|\alpha|^2\,|\beta|^2} \Big\{ \sum_{k=\pm} k\, \breve{s}^k \widetilde{\mathcal{W}}(\breve{a}^k,\mathcal{I},\beta,h) + \sum_{l=\pm} l\, t^l \widetilde{\mathcal{W}}(b^l,\mathcal{I},\breve{\alpha},h) \Big\}, \end{aligned} \tag{58}$$

where $\mathcal{R}$ and $\mathcal{S}$ are given by (71) and (76). The definitions of $\widetilde{\mathcal{V}}$ and $\widetilde{\mathcal{W}}$ follow

$$\widetilde{\mathcal{V}}(\breve{a},\mathcal{I},\beta,h) = \int_\beta (y-\mathcal{I}|\overrightarrow{\beta}) f_2(d,\|\breve{a}-y\|) d\gamma_y, \tag{59}$$

$$\text{and } \widetilde{\mathcal{W}}(\breve{a},\mathcal{I},\beta,h) = \int_\beta (y-\mathcal{I}|\overrightarrow{\beta}) f_3(d,\|\breve{a}-y\|) d\gamma_y, \tag{60}$$

with f_3 defined by (52). Explicit formulas for these two functions are given by (86) and (87).

4.2.3 The case of coplanar triangles

The reduction of $P^{\phi_q}(a,T)$ uses formula (5) with a as origin (see Figure (5):

$$P^{\phi_q}(a,T) = \phi_q(a)\sum_{m=1}^{3}\frac{\delta_m(a)}{2}R(a,\beta_m) + \frac{\delta_{q+1}(a)}{2}R^{\phi_q}(a,\beta_{q+1}) + \frac{\delta_{q+2}(a)}{2}R^{\phi_q}(a,\beta_{q+2}) \quad (61)$$

with R and R^{ϕ} respectively given by (70) and (75).

Segments with secant supports For these cases, we choose the intersection $\mathcal{I}$ of the supports of α and β as origin (see Figure 6(a)).

Calculation of $Q^{1,\phi_q}(\alpha,\beta)$

$$\begin{aligned} Q^{1,\phi_q}(\alpha,\beta) = \frac{\phi_q(\underline{o})}{2}\left\{\sum_{k=\pm} k\, s^k R(a^k,\beta) + \sum_{l=\pm} l\, t^l R(b^l,\alpha)\right\} \\ + \sum_{k=\pm} k\, s^k R^{\phi_q}(a^k,\beta) + \sum_{l=\pm} l\, t^l \phi_q(b^l) R(b^l,\alpha), \end{aligned} \quad (62)$$

with $\phi_q(b^l)$ which vanishes at one of the ends of β.

Calculation of $Q^{\phi_p,\phi_q}(\alpha,\beta)$ Using a similar decomposition as in (56) and (57), we have

$$\begin{aligned} \phi_p(x) = C_\alpha + \mu\frac{(x-\mathcal{I}|\overrightarrow{\alpha})}{|\alpha|^2} \text{ and } \phi_q(y) = C_\beta + \nu\frac{(y-\mathcal{I}|\overrightarrow{\beta})}{|\beta|^2}, \\ \text{with } \mu = 1 \text{ if } a_p = a^+, -1 \text{ if } a_p = a^- \,;\, \nu = 1 \text{ if } b_q = b^+, -1 \text{ if } b_q = b^-, \end{aligned} \quad (63)$$

and $C_\alpha = 1 + \frac{\mu(\mathcal{I}-a_p|\overrightarrow{\alpha})}{|\alpha|^2}$ and $C_\beta = 1 + \frac{\nu(\mathcal{I}-b_q|\overrightarrow{\beta})}{|\beta|^2}$. By formula (5), one has

$$\begin{aligned} & Q^{\phi_p,\phi_q}(\alpha,\beta) = \\ & \sum_{k=\pm} k\,\underline{s}^k\left[\left(C_\alpha C_\beta + \frac{\mu C_\beta}{2|\alpha|^2}\right)R(a^k,\beta) + \left(\frac{\nu C_\alpha}{2|\beta|^2} + \frac{\mu\nu}{3|\alpha|^2|\beta|^2}(a^k-\mathcal{I}|\overrightarrow{\alpha})\right)\mathcal{V}(a^k,\mathcal{I},\beta)\right] \\ & + \sum_{l=\pm} l\,\underline{t}^l\left[\left(C_\alpha C_\beta + \frac{\nu C_\alpha}{2|\beta|^2}\right)R(b^l,\alpha) + \left(\frac{\mu C_\beta}{2|\alpha|^2} + \frac{\mu\nu}{3|\alpha|^2|\beta|^2}(b^l-\mathcal{I}|\overrightarrow{\beta})\right)\mathcal{V}(b^l,\mathcal{I},\alpha)\right], \end{aligned} \quad (64)$$

with R given by (70) and

$$\mathcal{V}(a,\mathcal{I},\beta) = \int_\beta \frac{(y-\mathcal{I}|\overrightarrow{\beta})}{\|y-a\|}d\gamma_y. \quad (65)$$

The final expression of $\mathcal{V}$ is given by (85).

Segments with parallel supports When supports are parallel (see Figure 6(b)), we must use formula (4) instead of (5).

Calculation of $Q^{1,\phi_q}(\alpha,\beta)$ As the only basis function is relative to β, it is on the support of β that we will project α. Choosing $\underline{o} = b^-$ we obtain

$$\begin{aligned}Q^{1,\phi_q}(\alpha,\beta) &= \int_{\widetilde{\alpha}\times\beta} \frac{\phi_q(y)}{\sqrt{d^2+\|\widetilde{x}-y\|^2}} d\gamma_{\widetilde{x}} d\gamma_y \\ &= \phi_q(b^-)\left\{|\beta|\mathcal{T}\left(b^+,\widetilde{\alpha},d\right) + \sum_{k=\pm} k\,\underline{s}^k \mathcal{T}\left(\widetilde{a}^k,\beta,d\right)\right\} \\ &+ |\beta|\phi_s(b^+)\mathcal{S}(b^+,\widetilde{\alpha},d) + \sum_{k=\pm} k\,\underline{s}^k \mathcal{S}^{\phi_q}(\widetilde{a}^k,\beta,d).\end{aligned}$$

The expressions of $\mathcal{S}, \mathcal{S}^\phi$ and $\mathcal{T}$ are respectively given by (76),(81) and (83).

Calculation of $Q^{\phi_p,\phi_q}(\alpha,\beta)$ We use a similar decomposition as in (56) and (57). This time, we replace $\mathcal{I}$ by a^- which will be the new origin and we project β on the support of α. We denote by $\widetilde{\beta}$ and $\widetilde{y}$ the respective projections of β and y on the support of α and $d = \|b^- - \widetilde{b}^-\|$.

$$\phi_p(x) = 1 + \mu\frac{(x-a_p|\overrightarrow{\alpha})}{|\alpha|^2} = C_\alpha + \mu\frac{(x-a^-|\overrightarrow{\alpha})}{|\alpha|^2} \tag{66}$$

$$\phi_q(y) = 1 + \nu\frac{(y-b_q|\overrightarrow{\beta})}{|\beta|^2} = C_\beta + \nu\frac{(\widetilde{y}-a^-|\overrightarrow{\beta})}{|\beta|^2} \tag{67}$$

with $\mu = 1$ if $a_p = a^+, -1$ if $a_p = a^-$; $\nu = 1$ if $b_q = b^+, -1$ if $b_q = b^-$

and $C_\alpha = 1 + \frac{\mu(a^- - a_p|\overrightarrow{\alpha})}{|\alpha|^2}, C_\beta = 1 + \frac{\nu(a^- - \widetilde{b}_q|\overrightarrow{\beta})}{|\beta|^2}$. After some calculations, one has

$$Q^{\phi_p,\phi_q}(\alpha,\beta) = \int_{\alpha\times\beta} \frac{\phi_p(x)\phi_q(y)}{\|x-y\|} d\gamma_x d\gamma_y \tag{68}$$

$$\begin{aligned}&|\alpha|\left[C_\alpha C_\beta \mathcal{R}(a^+,\widetilde{\beta},d) + \mu C_\beta \mathcal{S}(a^+,\widetilde{\beta},d) + \frac{\nu C_\alpha}{|\beta|^2}\widetilde{\mathcal{V}}(a^+,a^-,\widetilde{\beta},d) + \frac{\mu\nu}{|\beta|^2}\widetilde{\mathcal{W}}(a^+,a^-,\widetilde{\beta},d)\right] \\ &+ \sum_{l=\pm} l\,\underline{t}^l \left[C_\alpha C_\beta \mathcal{R}(\widetilde{b}^l,\alpha,d) + \frac{\nu C_\alpha}{|\beta|^2}(\widetilde{b}^l - a^-|\overrightarrow{\beta})\mathcal{S}(\widetilde{b}^l,\alpha,d)\right] \\ &+ \sum_{l=\pm} l\,\underline{t}^l \left[\frac{\mu C_\beta}{|\alpha|^2}\widetilde{\mathcal{V}}(\widetilde{b}^l,a^-,\alpha,d) + \frac{\mu\nu}{|\alpha|^2|\beta|^2}(\widetilde{b}^l - a^-|\overrightarrow{\beta})\widetilde{\mathcal{W}}(\widetilde{b}^l,a^-,\alpha,d)\right],\end{aligned}$$

with $\mathcal{R}, \mathcal{S}, \widetilde{\mathcal{V}}$ and $\widetilde{\mathcal{W}}$ given. We emphasize that a^+ and $\widetilde{b}^\pm$ belong to the support of α and $\widetilde{\beta}$.

5 CONCLUSION

We have reduced 4-D integrals to a linear combination of 1-D regular integrals which can be numerically or even explicitely evaluated. In this last case, a high degree of accuracy can be obtained, even in the case of nearly singular integrals. Despite some lengthy calculations, the principle is rather straightforward and the method is quite flexible.

Moreover, the case of integrals with linear basis functions is discussed briefly for identical and adjacent triangles. It is possible to use our method for Collocation method, 2-D BEM and volume integral equations as well as for other equations provided that the integrand is positively homogeneous. Only the singular part of the single layer is presented in this paper but we have formulas for the double layer with constant basis functions.

A CALCULATION OF 1-D INTEGRALS

A.1 Calculation of $R(b, \alpha)$

With the notations of Figure 2, defining the distance $d = \|b - p(b)\|$ and finally choosing $p(b)$ as origin, one has

$$R(b,\alpha) = \int_{\alpha} \frac{1}{\sqrt{d^2 + \|p(b) - x\|^2}} dx = \int_{s^-}^{s^+} \frac{1}{\sqrt{d^2+s^2}} ds \tag{69}$$

$$= \sum_{k=\pm} k \operatorname{arg\,sinh}\left(\frac{s^k}{d}\right). \tag{70}$$

Evaluation of $\mathcal{R}(a, \beta, h)$ When a does not belong on the support of β, we use $q(a)$ the projection of a on the support of β as origin.

$$\begin{aligned}\mathcal{R}(a,\beta,h) = \int_\beta f_1(h, \|x-a\|)dx &= \int_\beta \frac{\sqrt{h^2+d^2+\|x-\tilde{a}\|^2} - h}{(d^2 + \|x-\tilde{a}\|^2)} d\gamma_x \\ &= \left[\operatorname{arg\,sinh}\left(\frac{t}{\sqrt{d^2+h^2}}\right) - \frac{h}{d}\operatorname{arg\,tan}\left(\frac{t}{d}\right)\right. \\ &\left. + \frac{h}{d}\operatorname{Im}\left\{\operatorname{arg\,tanh}\left(\frac{h^2+d^2+itd}{h\sqrt{h^2+d^2+t^2}}\right) - \frac{i\pi}{2}\operatorname{sgn}(t)\right\}\right]_{t=\underline{t}^-}^{\underline{t}^+},\end{aligned} \tag{71}$$

with $d = \|a - q(a)\|$. In the case of coplanar triangles with parallelism between segments, a belongs to the support of β so $d = 0$ and the formula becomes

$$\mathcal{R}(a,\beta,h) = \left[\operatorname{arg\,sinh}\left(\frac{s}{h}\right) - \frac{\sqrt{s^2+h^2} - h}{s}\right]_{t=\underline{t}^-}^{\underline{t}^+} \tag{72}$$

Calculation of $R^{\phi_i}(b, \alpha)$ On the edge α_i, the basis function ϕ_i vanishes so we just do the calculation on $\alpha_{i+k}, k = 1, 2$. The expression of ϕ_i on $\alpha_{i+k}, k = 1, 2$ is

$$\phi_i(x)_{|\alpha_{i+k}} = 1 - \frac{|x - a_i|}{|\alpha_{i+k}|}, k = 1 \text{ or } 2,$$

then

$$R^{\phi_i}(b, \alpha_{i+k}) = R(b, \alpha_{i+k}) - \frac{1}{|\alpha_{i+k}|}\int_{\alpha_{i+k}} \frac{|x-a_i|}{\|x-b\|} dx. \tag{73}$$

By introducing ζ_i the abscissa of a_i (s^-_{i+k} or s^+_{i+k} depending on the orientation chosen), we have

$$R^{\phi_i}(b, \alpha_{i+k}) = R(b, \alpha_{i+k}) - \frac{\mu}{|\alpha_{i+k}|}\int_{s^-_{i+k}}^{s^+_{i+k}} \frac{\zeta_i - s}{\sqrt{\gamma_i^2(b) + s^2}} ds, \tag{74}$$

with $\mu = 1$ if $a_i = a^+, -1$ if $a_i = a^-$; which leads to the final formula

$$R^{\phi_i}(b, \alpha_{i+k}) = R(b, \alpha_{i+k}) + \frac{\mu}{|\alpha_{i+k}|}\left[\sqrt{\gamma_i^2(b) + s} - \zeta_i \operatorname{arg\,sinh}\left(\frac{s}{|\gamma_i(b)|}\right)\right]_{s=\underline{s}_i^-}^{\underline{s}_i^+}. \tag{75}$$

Calculation of $\mathcal{S}$

$$\mathcal{S}(\widehat{a},\beta,h) = \left[\frac{h^2+d^2}{2d^2}\operatorname{arg\,sinh}\left(\frac{t}{\sqrt{h^2+d^2}}\right) - \frac{th^2}{2d^2\sqrt{d^2+t^2}}\operatorname{arg\,sinh}\left(\frac{\sqrt{d^2+t^2}}{h}\right)\right]_{\underline{t}^-}^{\underline{t}^+}. \quad (76)$$

If $\widehat{a}$ belongs to the support of β, then $d = 0$ and

$$\mathcal{S}(\widehat{a},\beta,h) = \left[\frac{h^2+2t^2}{4t^2}\operatorname{arg\,sinh}\left(\frac{t}{h}\right) - \frac{\sqrt{t^2+h^2}}{4t}\right]_{\underline{t}^-}^{\underline{t}^+}. \quad (77)$$

Calculation of $\mathcal{S}^\phi$ We use the decomposition: $\phi_q(y)_{|\beta} = 1 - \frac{|y-b_q|}{|\beta|}$, so

$$\mathcal{S}^{\phi_q}(\widehat{a},\beta,h) = \mathcal{S}(\widehat{a},\beta,h) - \frac{1}{|\beta|}\int_\beta |y-b_q| f_2(h,\|\widehat{a}-y\|)d\gamma_y \quad (78)$$

$$= \mathcal{S}(\widehat{a},\beta,h) - \frac{\nu}{|\beta|}\int_{t^-}^{t^+}(\zeta_q - t)f_2(h,\sqrt{d^2+t^2}dt, \quad (79)$$

with $\nu = -1$ if $b_q = b^-$ and $\nu = +1$ if $b_q = b^+$.

$$\mathcal{S}^{\phi_q}(\widehat{a},\beta,h) = (1-\frac{\nu\,\zeta_q}{|\beta|})\mathcal{S}(\widehat{a},\beta,h) + \frac{\nu}{|\beta|}\int_{\underline{t}^-}^{\underline{t}^+} tf_2(h,\sqrt{d^2+t^2}dt, \quad (80)$$

finally, one has

$$\mathcal{S}^{\phi_q}(\widehat{a},\beta,h) = (1-\frac{\nu\,\zeta_q}{|\beta|})\mathcal{S}(\widehat{a},\beta,h) + \frac{\nu}{|\beta|}\left[\frac{h^2+d^2}{2d^2}\operatorname{arg\,sinh}\left(\frac{s}{\sqrt{h^2+d^2}}\right)\right.$$

$$\left. - \frac{sh^2}{2d^2\sqrt{d^2+s^2}}\operatorname{arg\,sinh}\left(\frac{\sqrt{d^2+s^2}}{h}\right)\right]_{\underline{t}^-}^{\underline{t}^+}. \quad (81)$$

Calculation of $\mathcal{T}$

$$\mathcal{T}(\widehat{a},\beta,h) = \left[\frac{sh^2}{2d^2\sqrt{d^2+s^2}}\operatorname{arg\,sinh}\left(\frac{\sqrt{d^2+s^2}}{h}\right) - \frac{h^2-d^2}{2d^2}\operatorname{arg\,sinh}\left(\frac{s}{\sqrt{h^2+d^2}}\right)\right. \quad (82)$$

$$\left. - \frac{h}{d}\operatorname{arg\,tan}\left(\frac{s}{d}\right) - \frac{h}{d}\operatorname{Im}\left\{\operatorname{arg\,tanh}\left(\frac{h^2+d^2+itd}{h\sqrt{h^2+d^2+t^2}}\right) - \frac{i\pi}{2}\operatorname{sgn}(t)\right\}\right]_{\underline{t}^-}^{\underline{t}^+}. \quad (83)$$

Calculation of $\mathcal{V}(a,\mathcal{I},\beta)$ We introduce $q(a)$ the projection of a on the support of β and $d = \|a - q(a)\|$ in the expression (65), thus

$$\mathcal{V}(a,\mathcal{I},\beta) = \int_\beta \frac{(y-q(a)|\overrightarrow{\beta})}{\sqrt{d^2+\|y-q(a)\|^2}}d\gamma_y + (q(a)-\mathcal{I}|\overrightarrow{\beta})R(a,\beta). \quad (84)$$

Choosing $q(a)$ as origin on the support of β, we obtain

$$\begin{aligned}\mathcal{V}(a,\mathcal{I},\beta) &= \int_\beta \frac{(y-q(a)|\overrightarrow{\beta})}{\sqrt{d^2+\|y-q(a)\|^2}}d\gamma_y + (q(a)-\mathcal{I}|\overrightarrow{\beta})R(a,\beta)\\ &= \left[|\beta|^2\sqrt{d^2+t^2} + \left(q(a)-\mathcal{I}|\overrightarrow{\beta}\right)\operatorname{arg\,sinh}\left(\frac{t}{d}\right)\right]_{t=\underline{t}^-}^{t=\underline{t}^+}. \end{aligned}\tag{85}$$

Calculation of $\widetilde{V}$ Let $q(\breve{a})$ the projection of $\breve{a}$ on the support of β and $d = \|\breve{a}-q(\breve{a})\|$. We choose $q(\breve{a})$ as origin, then

$$\begin{aligned}\widetilde{\mathcal{V}}(\breve{a},\mathcal{I},\beta,h) = \frac{|\beta|^2}{2}&\left[\frac{h^2\operatorname{arg\,sinh}\left(\frac{\sqrt{d^2+s^2}}{h}\right)}{\sqrt{d^2+s^2}} + \sqrt{h^2+d^2+s^2}\right.\\ &- \frac{h}{2}\operatorname{arg\,tanh}\left(\frac{h^2+d^2+isd}{h\sqrt{h^2+d^2+s^2}}\right) - \frac{h}{2}\operatorname{arg\,tanh}\left(\frac{h^2+d^2-isd}{h\sqrt{h^2+d^2+s^2}}\right)\\ &\left.+ h\operatorname{arg\,sinh}\left(\frac{h}{\sqrt{d^2+s^2}}\right)\right]_{t=\underline{t}^-}^{\underline{t}^+} + \left(q(\breve{a})-\mathcal{I}|\overrightarrow{\beta}\right)\mathcal{S}(\breve{a},\beta,h).\end{aligned}\tag{86}$$

When $\breve{a}$ belongs to the support of β then $d=0$. In this case, there is no difficulty in the expression (86).

Calculation of $\widetilde{W}$ Similarly to (86), one has

$$\begin{aligned}\widetilde{\mathcal{W}}(\breve{a},\mathcal{I},\beta,h) = &\left[\frac{|\beta|^2}{3}\frac{(h^2+d^2+t^2)^{3/2}-h^3}{d^2+t^2} + \frac{\left(q(\breve{a})-\mathcal{I}|\overrightarrow{\beta}\right)}{3}\left(\operatorname{arg\,sinh}\left(\frac{t}{\sqrt{h^2+d^2}}\right)\right.\right.\\ &+\frac{h^3}{d^3}\operatorname{arg\,tan}\left(\frac{t}{d}\right) - \frac{th^2(\sqrt{h^2+d^2+t^2}-h)}{d^2(d^2+t^2)}\\ &\left.\left.+\frac{h^3}{d^3}\mathrm{Im}\left\{\operatorname{arg\,tanh}\left(\frac{h^2+d^2+itd}{h\sqrt{h^2+d^2+t^2}}\right)-\frac{i\pi}{2}\operatorname{sgn}(t)\right\}\right)\right]_{t=t^-}^{t^+}.\end{aligned}\tag{87}$$

When $\breve{a}$ belongs to the support of β it is more complicated in (87) , we obtain the expression:

$$\begin{aligned}\widetilde{\mathcal{W}}(\breve{a},\mathcal{I},\beta,h) = &\left[\frac{|\beta|^2}{3}\frac{(h^2+t^2)^{3/2}-h^3}{t^2} + \frac{\left(\breve{a}-\mathcal{I}|\overrightarrow{\beta}\right)}{3}\left(\operatorname{arg\,sinh}\left(\frac{t}{h}\right)\right.\right.\\ &\left.\left.+\frac{h^2-t^2}{3s\sqrt{h^2+t^2}} - \frac{2h^3(\sqrt{h^2+t^2}-h)}{3t^3\sqrt{h^2+t^2}}\right)\right]_{t=t^-}^{t^+}.\end{aligned}\tag{88}$$

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