Exact evaluation of singular and near-singular integrals in Galerkin BEM

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ABSTRACT

The discretization of 3-D scattering problems by variational boundary element methods leads to the evaluation of such elementary integrals (see [4],[2] and [5]) as

$$\int_{S\times T} G(x,y)\ v(x)\, w(y)\ dx\, dy \text{ and } \int_{S\times T} \frac{\partial}{\partial n_y} G(x,y)\ v(x)\, w(y)\ dx\, dy \tag{1}$$

where v and w are polynomial basis functions, G is the Green kernel and S and T two planar polygons from the discretization of the boundary. Due to the singularity of the kernel, the numerical evaluation of these integrals may lead to inaccurate results when S and T are close to each other. We split G and its gradient into a regular part which involves classical numerical techniques and a singular part subject to our method. This new method consists in integrating exactly integrals such as

$$I = \int_{S\times T} \frac{1}{\|x-y\|}\, dx\, dy \text{ and } J_\zeta = \int_{S\times T} \frac{x-y}{\|x-y\|^{1+\zeta}}\, dx\, dy,\ \zeta \in \{0,2\}. \tag{2}$$

Basic formulas Let $f(x,d)\ :\Omega \subset \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$ a positively homogeneous function of degree q. We denote $I(d) = \int_\Omega f(z,d)\, dz$ which by Euler's formula and Green's theorem satisfies the differential equation (see [1] and [3]) :

$$(q+n)\, I(d) = dI'(d) + \int_{\partial\Omega} (\overrightarrow{z}\,|\overrightarrow{\nu}\,)\, f(z,d)\, d\gamma_z \tag{3}$$

where $\overrightarrow{\nu}$ is the exterior normal to Ω and $(\cdot|\cdot)$ is the scalar inner product. Provided $d^{-(q+n)} \int_\Omega f(z,d)\, dz \to 0$ as $d \to +\infty$ one obtains

$$I(d) = d^{q+n} \int_{\partial\Omega} (\overrightarrow{z}\,|\overrightarrow{\nu}\,) \int_d^{+\infty} \frac{f(z,t)}{t^{q+n+1}} dt d\gamma_z. \tag{4}$$

When $f(z,d)$ does not depend on d and $q+n \neq 0$ then

$$I = \frac{1}{q+n} \int_{\partial\Omega} (\overrightarrow{z}\,|\overrightarrow{\nu}\,)\, f(z)\, d\gamma_z. \tag{5}$$

As long as the inner integral in (4) can be explicitely evaluated, both formulas reduce an n-dimensional integral to an $(n-1)$ one. When Ω is an n-dimensional polyhedron (such as $S \times T$ with $n = 4$), $(\overrightarrow{z}\,|\overrightarrow{\nu}\,)$ is constant on each$(n-1)$-face of Ω, a simplification of crucial importance in the sequel.

The reduction process We have obtained formulas for three types of geometrical configurations : S and T are (i) coplanar, (ii) in secant planes and (iii) in parallel planes. All these cases are treated using formulas (5) or (4) or both, depending on the relative positions of S and T.
As an example, we present the simple but significant result for the self-influence coefficient ($S = T$). Let A_i be a vertex of the triangle, α_i the opposite side, γ_i their mutual distance and λ_i the exterior normal along α_i (see Figure 1). After 3 successive reductions using formula (5), one obtains

$$I = \int_{S\times S} \frac{1}{\|x-y\|} dxdy = \frac{2\,|S|}{3} \sum_{i=1,3} \gamma_i R\left(A_i, \alpha_i\right), \qquad (6)$$

where $R(A_i, \alpha_i)$ is a 1-D regular integral and is given if analytically by

$$R\left(A_i, \alpha_i\right) = \int_{\alpha_i} \frac{1}{\|A_i - y\|} dy = \operatorname{arg\,sinh} \frac{s_i^+ - \sigma_i}{\gamma_i} - \operatorname{arg\,sinh} \frac{s_i^- - \sigma_i}{\gamma_i} \qquad (7)$$

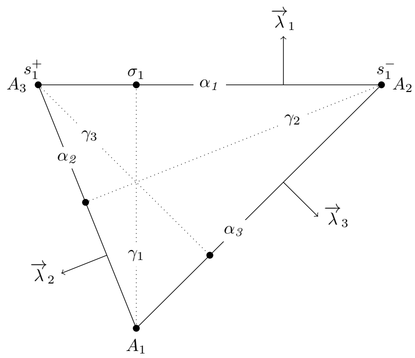

Figure 1: Notations

Results for the 3-D Helmholtz equation with piecewise constant density have been obtained for all pairs of panels. The extension to linear basis functions is in progress. Our method works also for 3-D Maxwell's equations with linear edge basis functions (MFIE and EFIE).

Conclusion Despite some (possibly) lengthy calculations, the principle is rather straightforward and the method is quite flexible, leading to the reduction of 4-D integrals to a linear combination 1-D regular integrals which can be numerically or even explicitely evaluated. It is possible to use our method for Collocation method, 2-D BEM and volume integral equations. A high degree of accuracy can be obtained, even in the case of nearly singular integrals. We will present the method and some results for 3-D Helmholtz equation.

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